The formula triangle and other problems with procedural teaching in mathematics

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ABSTRACT Students often express dislike of mathematics, even when they seem competent at it. They are often taught shortcuts for untangling mathematical problems; however, these shortcuts can bypass understanding and diminish a student's ability to recognise when an answer looks correct and when it does not. Using the examples of the formula triangle and a method for subtraction, it is shown that the mathematical steps to solve problems must be thoroughly understood before they are used, to lead to the understanding of short routines. Only then will students become confident in their answers.

A lot of students do not enjoy mathematics. Often, those that do are labelled as having a natural ability in the subject, but could it be simply that they have understood the elements that are implied but not explicitly taught? Is mathematics disliked so vocally because it is not applicable to everyday life? If so, surely the same could be said about most subjects at school. I am not sure when I last *needed* to recall anything about tectonic plates, for example. Perhaps, then, the truth is that, to many students, mathematics just does not make sense, and perhaps that is our fault.

In many mathematics lessons, students follow the rules we give them, calculate an answer we check for them and smile when we tell them it is right (or they sigh when it is not). I am not convinced that they *know* it is right, and often students are unable to reason that it is. It is not particularly surprising, then, that one of the most common phrases I have heard at parents' evenings is that, in mathematics, the problem is that 'your child is really lacking in *confidence*'. It is a familiar if perhaps misguided diagnosis of children who often get good grades but continually need reassurance that answers are correct.

A lack of confidence in itself should be an alarm bell that the underlying root of the problem is not confidence at all, it is *understanding*. If a student is getting all the right answers but needs reassurance, it is most likely because they simply do not know whether the answer is right or not.

Understanding does not always come with practising questions. It comes with great teaching. A student can practise finding the average (mean) of a set of numbers by adding them together and dividing them by how many there are for hours. It does not equate to them understanding *what* an average is, what it measures, or why we use it. Nor does it help them review their answer and get a feel for whether it looks right.

Take the operations of addition, subtraction, multiplication and division, for example. They are deceptively difficult to teach in mathematics. Deceptive in that on the surface it can look like a student has successfully managed to fully comprehend the concepts, decipher any given problems, and solve them correctly. Yet a little scratch at the surface can uncover a whole range of misconceptions and an inability to adapt to less prescribed questioning methods. These errors in understanding can lie dormant for years; in fact, they can go unnoticed for ever. Students with a generous capacity to remember facts and figures may be able to gain the best grades in mathematics without ever really comprehending it. But that does not bode well for the rest of us.

A student may be able to use the column method for addition a dozen times or more using the correct procedure and obtaining the right answers, but it does not mean they understand what they are doing. The difference can often be attributed to different teaching methodologies – broadly categorised into teaching procedurally and teaching conceptually. Procedural teaching provides students with a step-by-step guide to solving a problem, without reasoning. Conceptual teaching generally allows more scope to delve deeper into what is going on and why things work, to try to encourage an understanding and a level of reasoning alongside an ability to solve a given problem – but it takes time. The latter relies, of course, on the teacher possessing conceptual knowledge and understanding in the first place. However, one should not necessarily conclude that procedural teaching implies that a teacher does not have that knowledge and understanding.

For years, mathematics teaching in the UK, and indeed other Western countries, has been derided for a lack of conceptual teaching and an over-reliance on procedural methods. Ofsted itself has been particularly critical throughout the last decade of procedural teaching in mathematics, perhaps most explicitly in the 2012 document *Mathematics Made to Measure*, which contained the rather damning line:

While weak performance was generally challenged robustly, attention to the mathematical detail, so crucial in improving teachers' expertise, was lacking. (Ofsted, 2012: 7)

The recently published Sutton Trust report *What Makes Good Teaching?* (Coe, Aloisi, Higgins and Major, 2014) highlighted numerous research papers that have found certain teaching practices to be more effective than some recent popular methods. What is interesting is how criticisms of poor practice in mathematics are given substantially more attention than any other subject.

Mathematics Mastery Programmes

The latest strategies to discourage procedural teaching seem to be the Mathematics Mastery Programmes being promoted by the recently created Maths Teaching Hubs across the UK, and the promotion of Chinese teaching strategies in a bid to raise standards in mathematics. Many UK mathematics teachers have been somewhat sceptical of borrowing Eastern methods to teach Western students, often citing cultural differences and fundamental differences in the education systems of the East as a whole. However, one only has to go back to the hugely influential work *Knowing and Teaching Elementary Mathematics* (Liping, 1999) to see that the knowledge and understanding of mathematics, and the way in which it is communicated to students, is often far more conceptual and rigorously structured in the East than in many Western schools. Liping's work highlighted some of the key differences between conceptual and procedural teaching of mathematics, and suggested that procedural teaching is often a result of a personal lack of deeper understanding of mathematics, and therefore an *inability* to teach conceptually. One should certainly not conclude from this that all Western teaching is procedural or that all procedural teaching is based on a lack of understanding in mathematics. The picture is a lot more complicated than that. However, what is clear is that procedural teaching provides students with a very limited, inflexible environment within which to use mathematics.

A primary school example of procedural teaching

Let us consider a very simple example, one that would be taught in primary school: the subtraction of 32 from 191 using the column method (Figure 1):

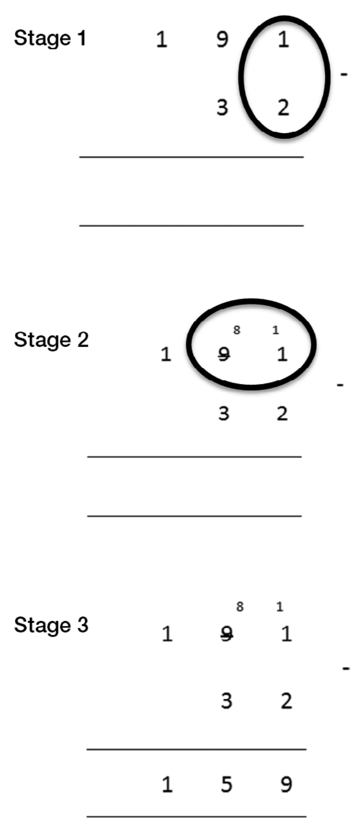

Figure 1 Illustration of a method used for subtraction at primary level

At stage 1, a common description associated with problems such as these is that you '*cannot* subtract 2 from 1'. Yet later we also teach students that $1-2=-1$, so we *can* subtract 2 from 1; however, in this method, we do not. But why? What are students learning when we tell them they cannot subtract 2 from 1? How will they feel about mathematics when we later tell them that $1-2=-1$? Are we not seeing the beginnings of mathematical confusion, of a world where mathematics is a mysterious, unpredictable thing that behaves as it wants to, and, to tame it, we simply need to memorise all of its quirks and anomalies? Suddenly, even with the knowledge that 32 is less than 191 (and therefore that $191-32$ should be straightforward), the student is beginning to find '*cannot*' and '*won't work*' where they were not expecting them. And to add to the confusion, in stage 2 of the calculation, students are often told they must now 'borrow 1 from the 9'. And so the 9 and the 1 are starting to be portrayed as separate somehow, rather than integrated. Furthermore, we are giving the impression that you can simply change parts of the sum around, borrowing what we like from one number and putting it with another. Can I also borrow from other parts of the number? Can I borrow from the 32? Without deeper exploration, and *explanation*, we are, right at the beginning of our mathematical adventure, becoming lost.

What is needed, particularly at these early beginnings, is an appreciation of the decimal system and its limitations, an exploration into how a number is made up, what each digit represents, how a number can be distributed into different parts but remain equal (191 is 1 hundred, 9 tens and 1 one, but it is also equal to 1 hundred, 8 tens and 11 ones, or 0 hundreds, 19 tens and 1 one, etc). This may seem advanced but in the East so much more time is spent on these concepts before moving on to formal addition methods, and so much time is spent on understanding how numbers behave, that students are not starting on the wrong footing.

A secondary school example of procedural teaching – the formula triangle

Further on in schooling, students will inevitably be introduced to the formula triangle. For the uninitiated, this is a visual prompt that allows for the rearranging of a simple formula that must be in the form $a=bc$ which bypasses any prerequisite knowledge of algebra. By placing a finger over the part of the triangle that you wish to be the subject, the remaining letters on show are easy to identify in the correct format as if it had been rearranged. Fortunately, there are a number of formulae that students use at school that come in the form $a=bc$, such as:

distance = speed × time

force = mass × acceleration

mass = density × volume

The formula triangle usually looks something like the example shown in Figure 2. However, it might be better understood if shown as in Figure 3.

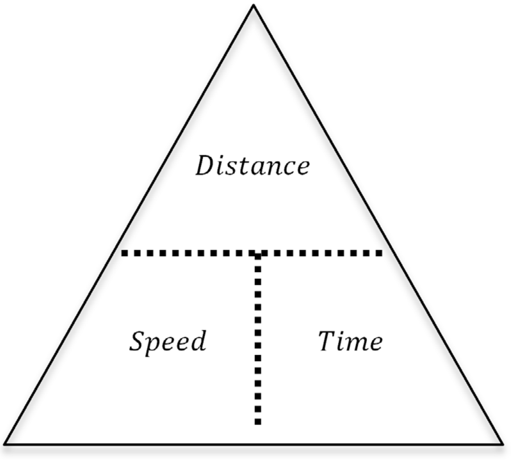

Figure 2 A typical layout of the 'formula triangle'

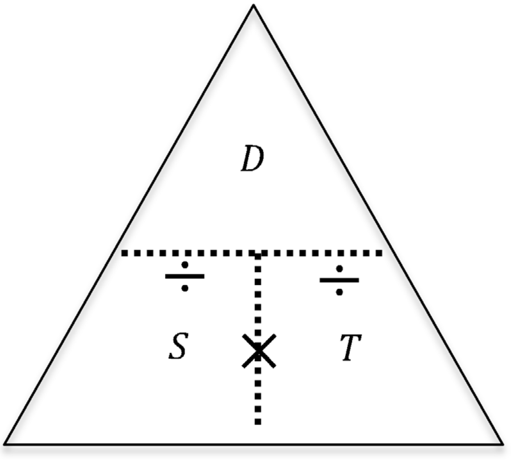

Figure 3 A formula triangle with mathematical symbols added

This approach is a classic procedural method. There is no mathematical reasoning behind what is being described, and the expectation is that students will simply mimic the steps and reach an answer. Questioning a student as to whether their answer is correct or not will often reveal the level of understanding they have. How would they know? Are they able to reverse engineer the question? Does the answer *look* right? All of these important elements of problem solving may be negated in exchange for a quick win. It is unsatisfying and inflexible. As with the previous example, it is supplying us with a correct answer but with no knowledge of *why* the answer is correct, or confidence that it is. There is the very real possibility that students may begin to erroneously apply the idea of a formula triangle to other equations. Consider $y=mx+c$. Rearranging for c is easier than rearranging speed $= \frac{\text{distance}}{\text{time}}$, but only if you have an understanding of what you are doing. Use a formula triangle and students will simply get the wrong answer. Worse still, a student could very realistically have no idea that it is even wrong.

The formula triangle is misleadingly simple. As a result, it is often called upon by students in situations where it is an inhibition rather than a convenient bypass of mathematics. An example often seen at GCSE (age 16) is a student incorrectly applying a formula triangle in trigonometry. The formula $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ is used to find either an angle θ or a side length of a right-angled triangle. When added to a formula triangle correctly, it looks as shown in Figure 4.

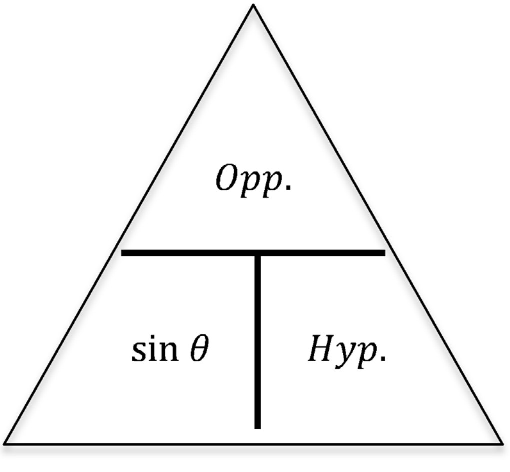

Figure 4 A formula triangle as needed if used to illustrate the definition of sine of an angle

Predictably, either students erroneously put $\sin\theta$ at the apex or they often decide the question is answered once they rearrange for $\sin\theta$ when, in fact, they have still to find the angle θ. The solution is incomplete, yet the students either do not notice, or are incapable of taking the answer any further as the triangle has exhausted its use.

Should the formula triangle be banned? No. It can be a useful method once fully understood and recognised as a shortcut. But it should not be derived from less efficient manipulation of algebra and it should not be the first port of call for teachers to help students get to an answer. These examples of shortcuts were derived because they are efficient. However, without appreciating the mathematical concepts behind them, students are being short-changed and will be ill prepared to handle more complex equations and tasks that require manipulation of them.

It may seem odd that, despite some clear disadvantages to teaching procedurally, many teachers still continue to do so. There are a multitude of possible explanations and, indeed, many suggestions have been offered in recent years. A particularly significant study by Malcolm Swan (2006) suggested that pressure to cover the curriculum, pressure to get good assessment scores quickly and difficulty in being fully prepared for the direction a lesson could go in could all contribute. It could also be that teachers are simply teaching mathematics in the way in which it was taught to them as students.

Is there change on the horizon?

The cycle of procedural teaching may finally be breaking. It seems that recent emphasis on mastery and a slower pace through mathematics may be having a positive effect on mathematics education. In 2014, King Solomon Academy, a comprehensive secondary school in a deprived area of London, received its first set of GCSE results after 5 years of students being taught using a rigorous Mathematics Mastery Programme. They achieved 95% A*–C grades in mathematics: 75% were a grade B+ and 40% were a grade A+. Furthermore, they did it again in 2015, this time with 55% grade A+. What is perhaps even more astounding is that 75% of King Solomon students went on to take mathematics at A-level.

Of course, there are numerous factors that will have contributed to this outstanding success story but the fact remains that teaching students for understanding has produced very talented mathematicians who are enthused enough about the subject to overwhelmingly continue studying it.

We may never see an end to procedural teaching of mathematics, and perhaps we should not wish for it to disappear *completely*. Granted, there will be students who will rely on procedural mathematics to get through their examinations, whether exposed to conceptual understanding or not. Perhaps therein lies a problem with the examination system itself. But to not even provide access to real understanding – to intentionally *avoid* it – is unfair to students.

Encouraging mystery over simplicity, fuzziness over clarity and an over-reliance on memorisation rather than an appreciation of the simple and logical laws of mathematics does not prepare students for further studies.

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LoMiS launch

The ASE's Language of Mathematics in Science project, funded by the Nuffield Foundation, will provide support for teachers through two publications, which will be available for download as PDF files.

The first booklet, *Language of Mathematics in Science: A Guide for Teachers of 11–16 Science*, will be available from mid-March at www.ase.org.uk/resources/maths-in-science.

See the article on page 15 of this issue for further details about these publications.