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Transient Stage Comparison of Couette Flow under Step Shear Stress and Step Velocity Boundary Conditions

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Abstract: Couette flow has been widely used in many industrial and research processes, such as viscosity measurement. For the study on thixotropic viscosity, step-loading, which includes (1) step shear stress and (2) step velocity conditions, is widely used. Transient stages of Couette flow under both step wall shear stress and step wall velocity conditions were investigated. The relative coefficient of viscosity was proposed to reflect the transient process. Relative coefficients of viscosity, dimensionless velocities and dimensionless development times were derived and calculated numerically. This article quantifies the relative coefficients of viscosity as functions of dimensionless time and step ratios when the boundary is subjected to step changes. As expected, in the absence of step changes, the expressions reduce to being functions of dimensionless time. When step wall shear stresses are imposed, the relative coefficients of viscosity changes from the values of the step ratios to their steady-state value of 1. but With step-increasing wall velocities, the relative coefficients of viscosity decrease from positive infinity to 1. The relative coefficients of viscosity increase from negative infinity to 1 under the step-decreasing wall velocity condition. During the very initial stage, the relative coefficients of viscosity under step wall velocity conditions is further from 1 than the one under step wall shear stress conditions but the former reaches 1 faster. Dimensionless development times grow with the step ratio under the step-rising conditions and approaches the constant value of 1.785 under the step wall shear stress condition, and 0.537 under the step wall velocity condition respectively. The development times under the imposed step wall shear stress conditions are always larger than the same under the imposed step wall velocity conditions.

Key words: Couette flow; Boundary conditions; Step wall shear stress; Step wall velocity

Nomenclature

d	distance between the two plates, m
t	time, s
t^*	characteristic time, $t^* = d^2/\nu$, s
$\tilde{t}$	dimensionless time, $\tilde{t} = t/t^*$
t_d	development time, s

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$\tilde{t}_d$	dimensionless development time, $\tilde{t}_d = t_d/t^*$
u	fluid velocity, m/s
u_w	bottom wall velocity, m/s
$\tilde{u}$	dimensionless velocity distribution
u_s	steady velocity distribution of stage (1), m/s
u_d	velocity after the flow field has developed, m/s
u_0	bottom wall velocity at stage (1), m/s
u_1	bottom wall velocity at stage (2) , m/s
U	velocity during the developing period, m/s
x	vertical spatial coordinate, m
$\tilde{x}$	dimensionless vertical spatial coordinate
Greek Symbols	
β_n	eigen values
ε	step ratio
$\dot{\gamma}$	shear rate, s^{-1}
μ	dynamic viscosity of fluid, Pa·s
ρ	fluid density, kg/m^3
τ	shear stress, Pa
τ_w	bottom wall shear stress, Pa
τ_0	bottom wall shear stress at stage(1) , Pa
τ_1	bottom wall shear stress at stage(2) , Pa
τ_d	developed shear stress, Pa
ξ_n	eigen values
ν	kinematic viscosity of fluid, m^2/s
φ	relative coefficient of viscosity, dimensionless
φ_τ	φ under imposed wall shear stress condition, dimensionless
φ_u	φ under imposed wall velocity condition, dimensionless

1 INTRODUCTION

In fluid dynamics, Couette flow is the laminar flow of a viscous fluid in the space between two parallel plates, one of which is moving while the other one keeps static. The transient start-up of Couette flow between two parallel plates one of which moves suddenly under imposed constant velocity has been investigated by many researchers[1,2]. Bandyopadhyay and Mazumder [3] studied the longitudinal dispersion of passive contaminant in an incompressible fluid between a parallel plate channel oscillating wall velocities. Analytical solutions of transient Couette flows of Newtonian fluid with constant and time-dependent pressure gradients have been proposed by Mendiburu et al. [4]. But the boundary condition was always a constant velocity.

There have been other researches on unsteady Couette flow of non-Newtonian fluid or under special situations. Exact solutions of unsteady Couette flows of generalized Maxwell fluid with fractional derivative were obtained by Wang and Xu[5] and Tan et al.[6]. Theoretical analysis of the velocity field and stress field of generalized second order fluid was presented by Xu and Tan[7]. Analytic solutions for unsteady unidirectional flows of an incompressible second grade fluid with a free surface or a moving plate and the associated frictional forces were reported by Hayat et al.[8]. Unsteady Couette flows of a second grade fluid with space dependent viscosity, elasticity and density were investigated by Asghar et al.[9]. Velocity distributions for transient Couette flows of Oldroyed-B fluid between two infinite parallel plates have also been reported [10,11]. Lacaze et al.[12] presented solutions to describe the flows of a viscoplastic fluid subjected to constant and sinusoidal moving plate velocities in a relative wide cylindrical Couette device. Solutions for unsteady Couette flow through a porous medium or in a magnetic or electric field have also been reported [13-15]. Unsteady magnetohydrodynamics (MHD) Couette flow of a third-grade fluid in the presence of pressure gradient and Hall currents was discussed by Azram and Zaman [16]. Unsteady Couette flow through of a viscoelastic fluid through a parallel plate channel filled with a porous medium was studied by Attia et al.[17]. Makinde and Franks[18] studied transient reactive MHD Couette flow with temperature dependent viscosity and thermal conductivity. Boundary conditions of all above researches were either constant or time-dependent velocity at the moving wall. Imposed shear stress boundary conditions have never been considered.

There have been some researchers paying their attention on boundary conditions of controlled tangential surface force or torque. Ting[19] discussed unsteady Couette flows of second-order fluids with constant tangential surface force. Considering inertia of rotation, Ravey et al[20] studied the transient motions of rotor based on an air-bearing viscometer filled with Newtonian fluids. Transient Couette flow of an Oldroyd fluid between two circular cylinders was investigated by Bernardin and Nouar[21] . The flow was induced by applying a constant torque to the inner cylinder with the outer cylinder kept stationary. Asymptotic analysis on acceleration and relaxation process near the inner cylinder was carried out to determine the time for inner cylinder and fluid to reach steady state.

Muzychka and Yovanovich [22] investigated the transient stage of Couette flow between two parallel plates and proposed a compact expression to describe the time-dependent shear stress on the internal surface of moving plate. Emin Erdoğan[2] concluded that the time required to attain the asymptotic values of the skin friction or volume flux of a Couette flow is d^2/ν , where d indicates the gap between the two parallel plates and ν represents the kinematic viscosity of the fluid. There have been several reasonable time scales for Taylor vortex flow to reach fully developed in cylindrical Couette flow, such as $t^* = d^2/\nu$, where d indicates radial gap[23,24], $t^* = H^2/\nu$, where H indicates gap length[25, 26], and a mixed time scale based on both H and d[27, 28], $t^* = Hd/\nu$. Another description of viscous time scale incorporating both the gap width and the distance between the endwalls of the system was recommended by Czarny and Lueptow

[29].

Couette flow has been used as the fundamental method for the measurement of viscosity[22]. Viscosity of time-dependent viscous fluid, such as thixotropic fluid, needs to be measured and recorded continuously. Step-loadings, either step shear stress or step shear rate are widely used to study the viscosity of thixotropic fluid [30-33]. The velocity distribution is not steady immediately after step-changing of boundary conditions. Only when the velocity gradient of fluid becomes time-invariant, will the measured viscosity represents the actual one.

In this article, transient Couette flow of an incompressible Newtonian fluid between two infinite parallel plates subjected to both step wall shear stress and step wall velocity boundary conditions are analyzed. The times needed for the velocity fields to allow for proper viscosity measurements called the "development times" are presented. For quantitative analysis on transient stage, we define the relative coefficient of viscosity $\varphi(t)$ and dimensionless development time $\tilde{t}_d$. The potential errors on viscosity measurements under different loading conditions are quantified and compared by examining these two parameters.

The remainder of this article is divided into seven sections. The problem analyzed in this article is outlined in the next section. This is followed by the definitions of the dimensionless parameters used to identify the development times. Section 4 presents the governing equations, the initial condition and the boundary conditions for the problem described in Section 2. The velocity distributions are then listed in Section 5. The relative coefficient of viscosity and the dimensionless velocity are then obtained in Section 6. This is followed by discussion of the results. Some concluding remarks are given to conclude the article.

2 PROBLEM DESCRIPTION

When viscosity is measured in coaxial cylinder viscometer, fluid is kept in concentric cylindrical cylinders. One cylinder is kept stationary while the other rotates. This leads to an axisymmetric velocity distribution. Neglecting end effects, the velocity is a function of the radial coordinate. For the study on thixotropic viscosity, step-loading is widely used which include (1) step shear stress and (2) step shear rate conditions. In a rheometer, shear rate is imposed by setting the rotational speeds of the measuring system[34]. So imposed shear rate condition can be analyzed as the imposed velocity condition. The flow in the annulus can be modeled using flow between two infinite parallel plates as the ratio of the annulus gap between inner and outer cylinders to the radius of the inner cylinder is small. The remainder of this article presents analyses of flow between two flat plates used to calculate viscosity.

Fig.1 shows the schematic of the problems considered in this article. Couette flows between two parallel plates separated by a distance d are analyzed. Transient laminar flows of an incompressible Newtonian fluid are induced by imposing different conditions on the bottom wall while the top wall remains stationary. The fluid is restricted to flow parallel to the two plates leading to a velocity field which is a function of vertical spatial coordinate x and time t. The flow is steady under a specific wall shear stress τ_0 $(\tau_0 \neq 0)$ or velocity u_0 $(u_0 \neq 0)$ in stage (1) and the velocity distribution could be described by

$$u_s(x) = \frac{\tau_0}{\mu}(d - x) \tag{1}$$

under the imposed wall shear stress condition and

$$u_s(x) = \frac{u_0}{d}(d - x) \tag{2}$$

under the imposed wall velocity condition. Here u_s is the steady velocity of stage (1) and also the initial velocity of stage (2). At the beginning of stage (2), the wall shear stress jumps to τ_1 $(\tau_1 \neq \tau_0 > 0)$ or the wall velocity jumps to u_1 $(u_1 \neq u_0 > 0)$. Transient flows caused by the step boundary conditions are sought after in this article.

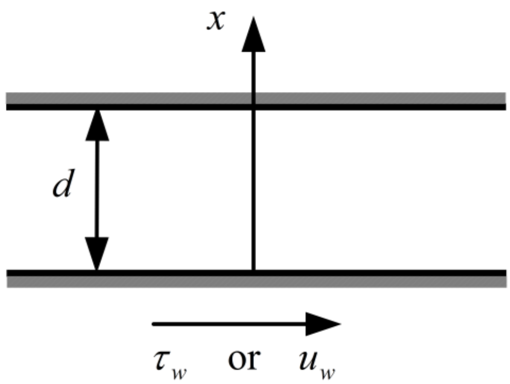


Fig.1 Schematic of the problem

3 BASIC DEFINITIONS

3.1 Relative Coefficient of Viscosity

There are two different ways to determine the fluid viscosity which include (1) imposing the shear stress and (2) imposing the shear rate. For a measuring system which is based on Couette flow shown in Fig.1, the shear rate $\dot{\gamma}$ is calculated from

$$\dot{\gamma} \equiv \frac{u_w}{d} = \frac{u(0,t)}{d} \tag{3}$$

where u_w is the wall velocity.

For a Newtonian fluid, the wall shear stress τ_w can be determined from

$$\tau_w \equiv \tau(0,t) = -\mu_a \frac{\partial u(0,t)}{\partial x} = \mu(t) \cdot \dot{\gamma} \tag{4}$$

where $\mu(t)$ is the calculated viscosity based on the measuring principle, μ_a is the actual dynamic viscosity of fluid and x is the distance normal to the plates (measured from the bottom wall) as shown in Fig.1. After an initial transient period where the velocity develops, the developed velocity $u(x,t)$ is linear in x and is related to the shear stress $\tau(x,t)$ by

$$\tau_d(0, t > t_d) = \mu_a \frac{u_d(0,t)}{d} = \mu_a \dot{\gamma}_d \tag{5}$$

where τ_d is the developed shear stress, t_d is the development time, $\dot{\gamma}_d$ is the developed shear rate. During the initial transient, the measured fluid viscosity can be different from the actual fluid viscosity. A relative coefficient of viscosity $\varphi(t)$ is defined as

$$\varphi(t) \equiv \frac{\mu(t)}{\mu_a} \tag{6}$$

Imposed wall shear stress $\boldsymbol{\tau_w}$

When wall shear stress is imposed, $\tau(0,t) = \tau_d(0,t) = \tau_w$, $\dot{\gamma}$ is calculated from the wall velocity. Using Eq. (3) to Eq. (6), the relative coefficient of viscosity is then calculated through

$$\varphi_\tau(t) = \frac{\tau(0,t)d/u(0,t)}{\tau_d(0,t)d/u_d(0,t)} = \frac{u_d(0,t)}{u(0,t)} \tag{7}$$

where the subscript τ denotes imposed wall shear stress.

Imposed wall velocity $\boldsymbol{u_w}$

When the shear rate is imposed, the velocity of moving boundary is specified. Based on the ideal Couette problem shown in Fig.1, there will be $u(0,t) = u_d(0,t) = \dot{\gamma}d$. Using Eqs.(4) and (5). The relative coefficient of viscosity can be evaluated as

$$\varphi_u(t) = \frac{-\mu_a \frac{\partial u(0,t)}{\partial x} \Big/ \frac{u_d(0,t)}{d}}{\mu_a} = \frac{-\partial u(0,t)/\partial x}{u_d(0,t)/d} \tag{8}$$

where the subscript u denotes imposed wall velocity.

3.2 Dimensionless Velocity

Here we define the dimensionless velocity under different boundary conditions as

$$\tilde{u}(x,t) = \frac{u(x,t)}{u_s(0)} \tag{9}$$

where $u_s(0)$ is $u_s(x)$ at $x = 0$. Using Eqs.(1) and (2), there are $u_s(0) = \tau_0 d/\mu$ under the imposed wall shear stress condition and $u_s(0) = u_0$ under the imposed wall velocity condition.

4 MATHEMATICAL MODEL

4.1 Governing Equation

The Navier-Stokes equation for the fluid velocity $u = u(x, t)$ can be written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{10}$$

where t is the time, x is the vertical spatial coordinate. Note that a velocity field which satisfies the momentum equation (Eq. 10) also satisfies the continuity equation. As a result, the continuity equation is not listed.

4.2 Initial Conditions

Imposed wall shear stress

For a step shear stress condition, the initial condition for $t = 0$ can be described as

$$u(x, 0) = u_s(x) = \frac{\tau_0}{\mu}(d - x), \qquad 0 \le x \le d \tag{11}$$

Imposed wall velocity

For a step-changing wall velocity condition, the initial condition can be described as

$$u(x, 0) = \frac{u_0}{d}(d - x), \qquad 0 \le x \le d \tag{12}$$

4.3 Boundary Conditions

In this article, the top wall is always kept stationary and can be written as

$$u(d, t) = 0, \qquad t > 0 \tag{13}$$

Imposed wall shear stress

When step-changing occurs, a constant shear stress τ_1 which is different with τ_0 is imposed on the lower wall suddenly. The boundary condition can be written as

$$\tau(0, t) = -\mu \frac{\partial u(0, t)}{\partial x} = \tau_1 \tag{14}$$

Imposed wall velocity

When step-changing occurs, a constant velocity u_1 which is different with u_0 is imposed on the lower wall suddenly. The boundary condition could be written as

$$u(0, t) = u_1 \tag{15}$$

5 SOLUTIONS OF THE VELOCITY DISTRIBUTIONS

5.1 Imposed Wall Shear Stress

It was observed experimentally that after an initial development time period, say $t = t_d$, the velocity can be described by

$$u(x, t > t_d) \equiv u_d(x, t) = \frac{\tau_1}{\mu}(d - x) \tag{16}$$

where u_d is the developed solution. We can express the velocity field as a superposition of the developing solution $U(x, t)$ and the developed solution as

$$u(x,t) = U(x,t) + u_d(x,t) \tag{17}$$

Using Eqs.(16) and (17), the solution can be written as

$$u(x,t) = U(x,t) + \frac{\tau_1}{\mu}(d-x) \tag{18}$$

Now we seek the developing solution $U(x,t)$ to find the velocity for the problem. Eq. (A8) provides the solution of $U(x,t)$. Using Eq. (A8) and Eq. (18), the final solution is

$$u(x,t) = -\frac{2(\tau_1-\tau_0)}{\mu d}\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\exp(-\nu\beta_n^2 t)\cos\beta_n x + \frac{\tau_1}{\mu}(d-x) \tag{19}$$

where β_n is given in Eq.(A9).

5.2 Imposed Wall Velocity

After an initial development time period say $t = t_d$, the velocity can be written as

$$u(x,t>t_d) \equiv u_d(x,t) = \frac{u_1}{\mu}(d-x) \tag{20}$$

Using Eqs. (17) and (20), the velocity could be written as

$$u(x,t) = U(x,t) + \frac{u_1}{d}(d-x) \tag{21}$$

According to the solution in Eq. (B4), the final solution can be written as

$$u(x,t) = -\frac{2(u_1-u_0)}{d}\sum_{n=1}^{\infty}\frac{1}{\xi_n}\exp(-\nu\xi_n^2 t)\sin\xi_n x + \frac{u_1}{d}(d-x) \tag{22}$$

where ξ_n is given in Eq. (B5).

6 SOLUTIONS OF $\boldsymbol{\varphi(t)}$ AND $\boldsymbol{\tilde{u}(x,t)}$

In the article, we use the relative coefficient of viscosity $\varphi(t)$ defined in Eq. (6) and the dimensionless velocity $\tilde{u}(x,t)$ defined in Eq. (9) to describe the transient state of the Couette flow process.

6.1 $\boldsymbol{\varphi(t)}$ and $\boldsymbol{\tilde{u}(x,t)}$ Under Imposed Wall Shear Stress Condition

Using Eqs.(1), (9), (19) and (A9), we can obtain the dimensionless velocity under the imposed step wall shear stress condition as

$$\tilde{u}(\tilde{x},\tilde{t}) = -\frac{8(\varepsilon_\tau-1)}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tilde{t}\right]\cos\frac{(2n+1)\pi}{2}\tilde{x} + \varepsilon_\tau(1-\tilde{x}) \tag{23}$$

where shear stress step ratio $\varepsilon_\tau = \tau_1/\tau_0$, dimensionless vertical spatial coordinate $\tilde{x} = x/d$, dimensionless time $\tilde{t} = \nu t/d^2$, among which . When $\varepsilon_\tau > 1$, it's a step-rising shear stress and when $\varepsilon_\tau < 1$, it's a step-decreasing shear stress.

Using Eqs. (7), (19) and (A9), we can describe the relative coefficient of viscosity under the imposed step wall shear stress condition by

$$\varphi_\tau(\tilde{t}) = \frac{1}{-\frac{8}{\pi^2}\left(1-\frac{1}{\varepsilon_\tau}\right)\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tilde{t}\right]+1} \tag{24}$$

From Eq.(24), we can get that the relative coefficient of viscosity is a function of dimensionless time $\tilde{t}$ and shear stress step ratio ε_τ. From $\sum_{n=0}^{\infty}[1/(2n+1)^2] = \pi^2/8$, we can find that $\varphi_\tau(t \to 0) = \varepsilon_\tau$. Based on $\lim_{\tilde{t}\to+\infty}\exp[-(2n+1)^2\pi^2\tilde{t}/4] = 0$, we have $\varphi_\tau(t \to +\infty) = 1$.

When Couette flow starts from rest under an imposed wall shear stress, we can have $\varepsilon_\tau \to \infty$. Then Eq. (24) can be written as

$$\varphi_\tau(\tilde{t}) = \frac{1}{-\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tilde{t}\right]+1} \tag{25}$$

Eq.(25) describes the relative coefficient of viscosity during the transient stage of Couette flow which starts suddenly from rest under an imposed wall shear stress. Obviously, $\varphi_\tau(\tilde{t})$ is only relevant to the dimensionless time $\tilde{t}$ when the fluid is initially stationary at $t = 0$.

When the wall shear stress is removed suddenly from a steady Couette flow, we can get $\varepsilon_\tau = 0$. And we also find that $\tau_d = u_d = 0$. So the definition of $\varphi_\tau(t)$ in Eq. (7) would lose its meaning. In actual operation of viscosity measurement, this corresponds to the situation where the measurement has ended and thus no meaningful solutions are obtained.

6.2 $\boldsymbol{\varphi(t)}$ and $\boldsymbol{\tilde{u}(x,t)}$ Under Imposed Wall Velocity Condition

According to Eqs. (2), (9), (22) and (B5), we can obtain the dimensionless velocity under the imposed step wall velocity condition as

$$\tilde{u}(\tilde{x},\tilde{t}) = -\frac{2(\varepsilon_u - 1)}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-n^2\pi^2\tilde{t})\sin n\pi\,\tilde{x} + \varepsilon_u(1-\tilde{x}) \tag{26}$$

where velocity step ratio $\varepsilon_u = u_1/u_0$. When $\varepsilon_u > 1$, it's a step-rising velocity and when $\varepsilon_u < 1$, it's a step-decreasing velocity. Using Eqs.(8) , (22) and (B5), we can describe the relative coefficient of viscosity under the imposed step wall velocity condition by

$$\varphi_u(\tilde{t}) = 1 + 2\left(1-\frac{1}{\varepsilon_u}\right)\sum_{n=1}^{\infty}\exp(-n^2\pi^2\tilde{t}) \tag{27}$$

From Eq.(27), we can find that the relative coefficient of viscosity is a function of dimensionless time $\tilde{t}$ and velocity step ratio ε_u. Since $\lim_{\tilde{t}\to 0^+}\sum_{n=1}^{\infty}\exp(-n^2\pi^2\tilde{t}) \to +\infty$, it follows that $\varphi_u(0^+) \to +\infty$ when $\varepsilon_u > 1$ and $\varphi_u(0^+) \to -\infty$ when $\varepsilon_u < 1$. According to

$\lim_{\tilde{t}\to+\infty} \sum_{n=1}^{\infty} \exp(-n^2\pi^2\tilde{t}) = 0$, we can write $\varphi_u(+\infty) = 1$.

When Couette flow starts from rest under an imposed wall velocity, we can have $\varepsilon_u \to \infty$. Then Eq.(27) can be written as

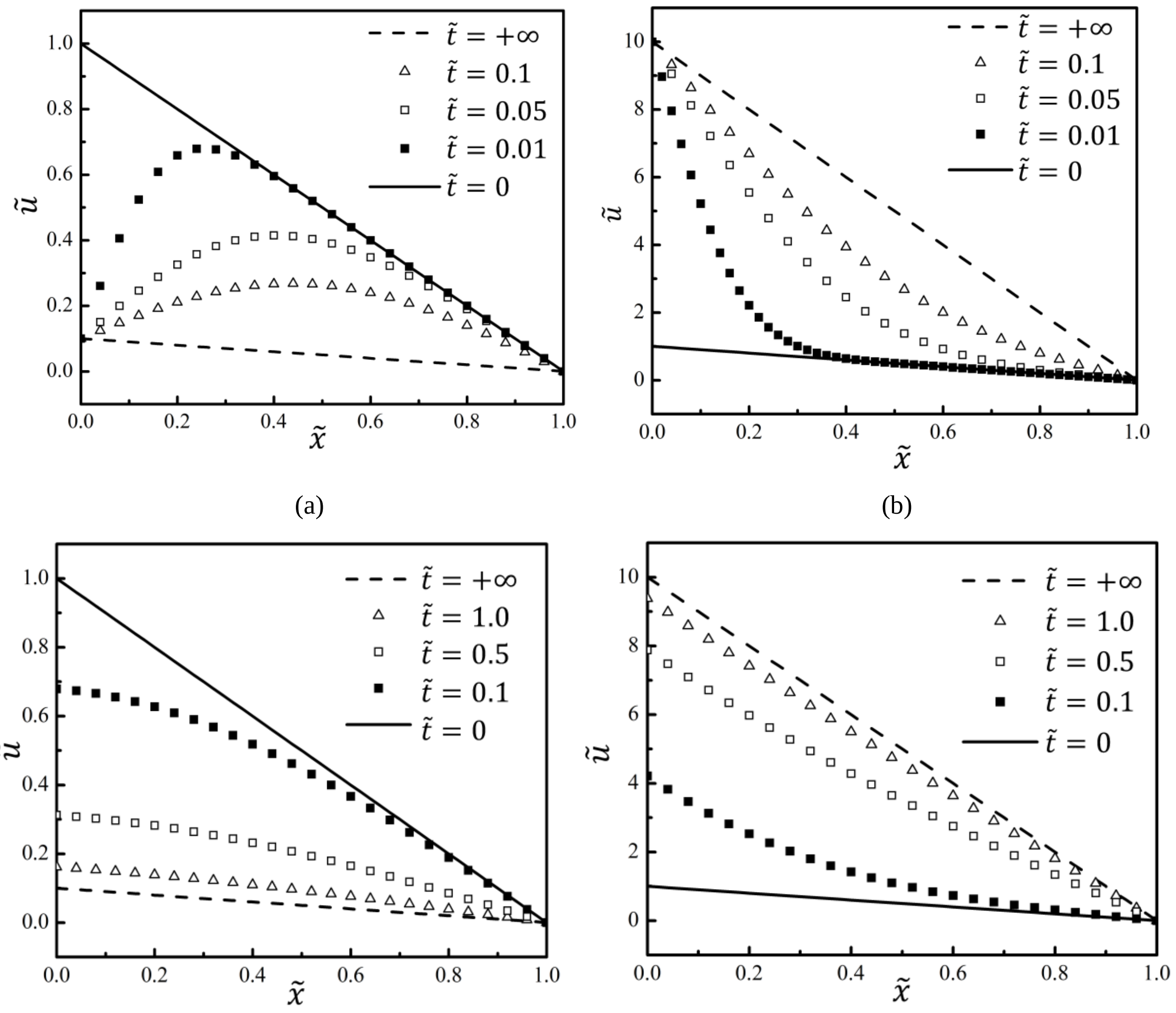
$$\varphi_u(\tilde{t}) = 1 + 2\sum_{n=1}^{\infty} \exp(-n^2\pi^2\tilde{t}) \tag{28}$$

Eq.(28) describes the relative coefficient of viscosity during the transient stage of Couette flow which starts suddenly from rest under an imposed wall velocity. Obviously, $\varphi_u(\tilde{t})$ is only relevant to $\tilde{t}$ when the fluid is initially stationary at $t = 0$. When the moving wall of steady Couette flow ceases suddenly, we can get $\varepsilon_u = 0$ and $u_d = 0$. The definition of $\varphi_u(t)$ in Eq.(8) loses its meaning and this situation won't be considered either.

7 RESULTS AND DISCUSSION

7.1 Comparison of $\tilde{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{t})$ under different boundary conditions

From Eqs. (23) and (26), we can find that the dimensionless velocity $\tilde{u}$ is a function of dimensionless time $\tilde{t}$, dimensionless vertical spatial coordinate $\tilde{x}$ and step ratios ε. Fig.2 shows the developments of the dimensionless velocities for step ratios of 0.1 and 10. The fluid velocity near the wall changes instantly under step velocity conditions and transient effect of changed velocity spreads along $\tilde{x}$ axial, so a significantly non-monotonic velocity profile which increases first and then decreases with $\tilde{x}$ occurs under the step-decreasing wall velocity condition (Fig.2a), but fluid velocity always decreases with $\tilde{x}$ under step-increasing velocity condition (Fig.2b). Under step wall shear stress conditions, velocity gradient near the wall changes immediately after shear stress steps and spreads along $\tilde{x}$ axial. So velocity of fluid changes gradually rather than instantly. The dimensionless velocity $\tilde{u}$ is always a monotone decreasing function of dimensionless vertical spatial coordinate $\tilde{x}$ under the step wall shear stress conditions (Fig.2c and Fig.2d). After a step-decreasing velocity happens, shear stress during the transient stage on the inner surface of moving wall may be opposite to the former steady one because of the non-monotonic velocity profile and the corresponding relative coefficient of viscosity $\varphi_u(t)$ is negative which is coincident with the conclusion obtained by Eq. (27).



(a) (b)

(c) (d)

Fig.2 Dimensionless velocity distributions under different step boundary conditions
(a)step-decreasing shear stress $\varepsilon_\tau = 0.1$; (b) step-increasing shear stress $\varepsilon_\tau = 10$;
(c) step-decreasing velocity $\varepsilon_u = 0.1$; (d) step-increasing velocity $\varepsilon_u = 10$

7.2 Comparison of $\boldsymbol{\varphi(t)}$ under different boundary conditions

Sudden step of wall velocity creates an infinite velocity gradient of fluid near the wall leading to an infinite $\varphi(t)$ based on Eq.(8). Different from the step velocity conditions, fluid velocity remains the same at the exact moment when wall shear stress steps and then changes gradually. So, no extremely large sudden changes occur in the fluid velocity and its gradient at the wall. As a result, $\varphi(t)$ is always finite. The relative coefficient of viscosity $\varphi(t)$ under different step boundary conditions is shown in Fig.3. From Fig.3(a), we can find that $\varphi(t)$ decreases versus time under the step-rising boundary condition. During the very initial stage, $\varphi_u(t)$ is always larger than $\varphi_\tau(t)$ because $\varphi_u(t)$ decrease from $+\infty$ but $\varphi_\tau(t)$ decrease from ε_τ. It can be seen that $\varphi_u(t)$ approaches 1 faster than $\varphi_\tau(t)$, so the development time t_d under the step-rising wall velocity condition is shorter than the one under the step-rising wall shear stress condition. After a development time t_d, $\varphi(t)$ equals 1 which means that the calculated viscosity is the actual viscosity of fluid. We will examine the development in more detail later in this article. Fig.3(b) shows the relative coefficient of viscosity $\varphi_u(t)$ under step wall velocity conditions. We can see that $\varphi_u(t)$ decreases from $+\infty$ to 1 when step-rising velocity occurs but increases from $-\infty$ to 1 under the step-decreasing velocity condition. Fig.3(c) shows the relative coefficient of viscosity $\varphi_\tau(t)$ under step wall shear stress conditions. We can see that $\varphi_\tau(t)$ decreases from ε_τ to 1 when $\varepsilon_\tau > 1$ but increases from ε_τ to 1 when $\varepsilon_\tau < 1$. From Fig.3(b) and Fig.3(c) we can find that the development time t_d for $\varphi(t)$ reaching 1 is slightly different with each other when the step ratio ε changes. So the development time will be discussed in the following part of the article.

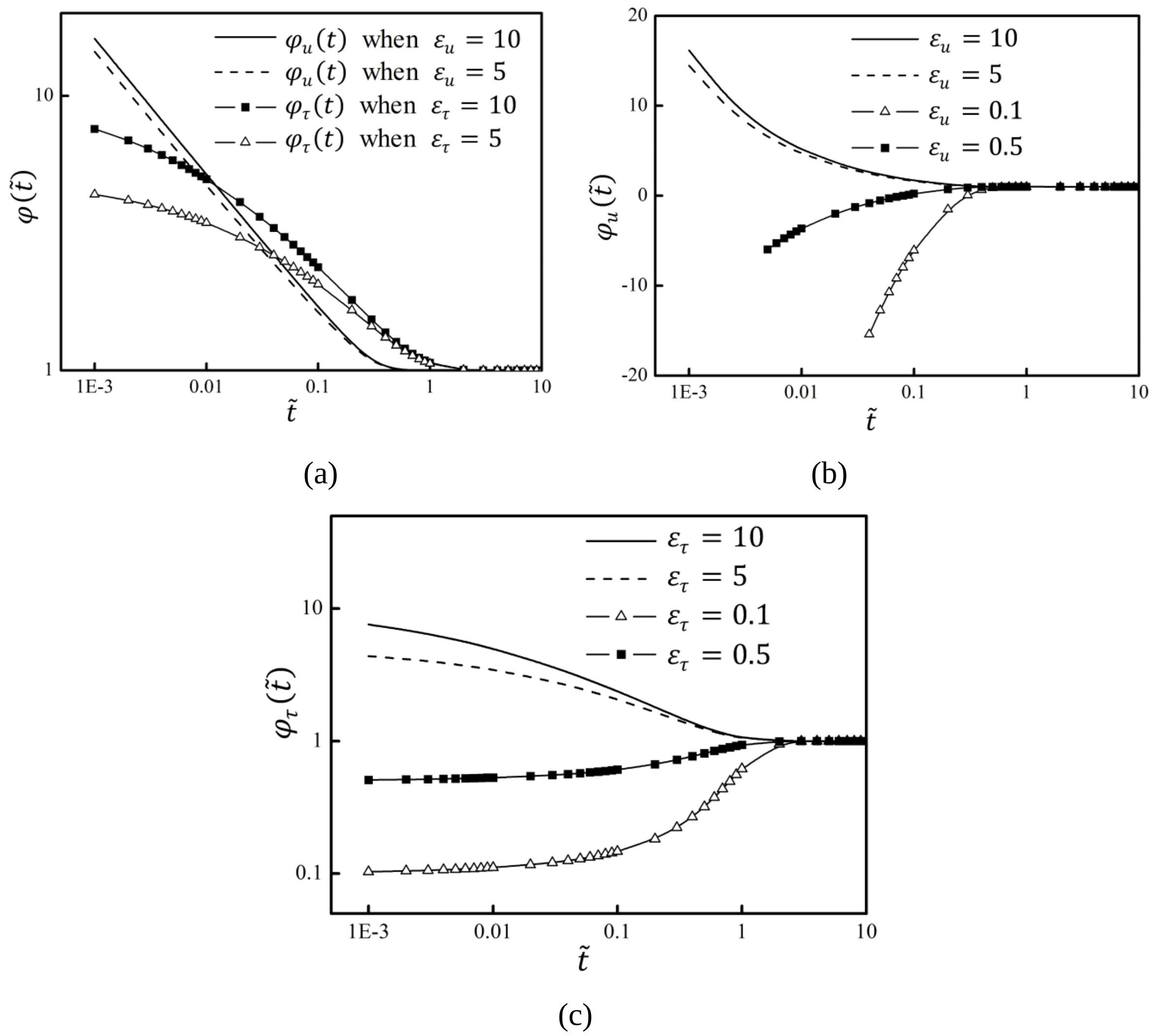


Fig.3 $\varphi(t)$ under different step conditions: (a) Comparison of $\varphi(t)$ between step-rising wall shear stress and velocity conditions; (b) $\varphi(t)$ under step-increasing and step-decreasing wall velocity conditions; (c) $\varphi(t)$ under step-increasing and step-decreasing wall shear stress conditions

7.3 Comparison of $\tilde{t}_d$ under different boundary conditions

The dimensionless development time $\tilde{t}_d$ is introduced to indicate the time period for the transient flow to reach steady state. For quantitative analysis, we define the developed state as

$$\frac{|\varphi(\tilde{t} \geq \tilde{t}_d) - \varphi(+\infty)|}{\varphi(+\infty)} \leq 1\% \tag{29}$$

where $\varphi(+\infty)$ indicates the relative coefficient of viscosity when $\tilde{t} \to +\infty$ so there is $\varphi(+\infty) = 1$. According to Eq. (29), $\tilde{t}_d$ could be described as

$$|\varphi(\tilde{t}_d) - 1| = 1\% \tag{30}$$

According to Eqs.(25) and (30), we can get that $\tilde{t}_d = 1.785$ under the imposed step wall shear stress condition when $\varepsilon_\tau \to \infty$. And from Eqs. (28) and (30), we can find that $\tilde{t}_d = 0.537$ under the imposed step wall velocity condition when $\varepsilon_u \to \infty$.

Based on Eqs. (24), (27) and (30), we calculated the dimensionless development time $\tilde{t}_d$ numerically and change laws of $\tilde{t}_d$ versus step ratio ε were shown in Fig.4. We can find that $\tilde{t}_d$ grows with ε when $\varepsilon > 1$ under both imposed step wall shear stress and velocity conditions. Under the imposed step wall shear stress condition, $\tilde{t}_d$ increases to a constant of 1.785 and under the imposed step wall velocity condition, $\tilde{t}_d$ increases to a constant of 0.537. So we believe that $\tilde{t}_d$ does not change with the step ratio ε when ε is large enough.

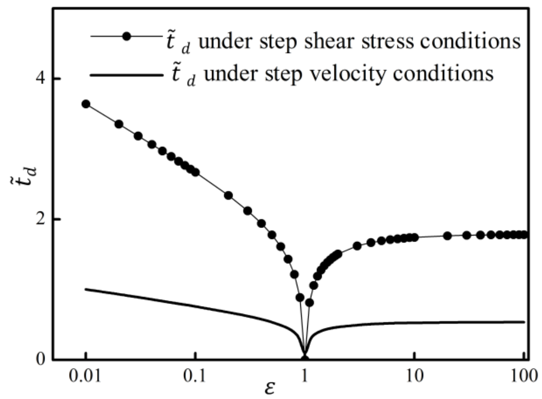


Fig.4 $\tilde{t}_d$ vs. step ratio ε under different boundary conditions

The dimensionless development time $\tilde{t}_d$ keeps increasing as the step ratio ε decreases and does not seem to be a constant value when $\varepsilon < 1$. We believe this is concerned with the definition of $\varphi(t)$ and criteria of transient flow being developed. It has been declared that definition of $\varphi(t)$ would lose its meaning when $\varepsilon = 0$, so the transient stage of Couette flow when $\varepsilon \to 0$ would be ignored here.

It has been found that fluid velocity changes gradually rather than instantly under step wall shear stress conditions. Slower velocity variation leads to longer time period for reaching steady state. From Fig.4 we can find that $\tilde{t}_d$ under the imposed step wall shear stress condition is always larger than the one under the imposed step wall velocity condition when a same step ratio is applied.

8 CONCLUDING REMARKS

In this article, the initial transient of Couette flows between two infinite parallel plates, subjected to both imposed step wall shear stress and step wall velocity boundary conditions were investigated. It was observed that

1. The dimensionless velocity $\tilde{u}$ is always a monotone decreasing function of $\tilde{x}$ under the step wall shear stress conditions and step-rising wall velocity condition. But under the step-decreasing velocity condition, $\tilde{u}$ increases first and then decreases with $\tilde{x}$ and the corresponding shear stress on the inner surface of moving wall is opposite to the former steady one.

2. Normally, $\varphi(t)$ is a function of dimensionless time $\tilde{t}$ and step ratio ε. When the fluid is initially stationary at $t = 0$, $\varphi(t)$ is only relevant to the dimensionless time $\tilde{t}$. $\varphi_\tau(t)$ changes from ε_τ to 1 gradually with $\tilde{t}$ under the step wall shear stress conditions. Under the step wall velocity conditions, $\varphi_u(t)$ decreases from $+\infty$ to 1 with $\tilde{t}$ when $\varepsilon_u > 1$ and increases from $-\infty$ to 1 with $\tilde{t}$ when $\varepsilon_u < 1$. $\varphi_u(t)$ is further from 1 than $\varphi_\tau(t)$ during the very initial stage but reaches 1 faster during the transient stage. Based on this and whenever possible, it is recommend that step change in shear rate be used experimentally to measure the fluid viscosity.

3. $\tilde{t}_d$ grows with ε when $\varepsilon > 1$ and approaches to the constant value of 1.785 under the step wall shear stress condition, 0.537 under the step wall velocity condition respectively. When $\varepsilon < 1$, $\tilde{t}_d$ keeps increasing as the step ratio ε decreases. $\tilde{t}_d$ under the imposed step wall shear stress condition is always larger than the one under the imposed step wall velocity condition.

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APPENDIX A

Based on Eq.(18), the spatial and temporal gradients of the solution are

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x} - \frac{\tau_1}{\mu} \tag{A1}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial x^2} \tag{A2}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} \tag{A3}$$

Using Eqs. (A2) and (A3), the governing equation Eq. (10) becomes

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial x^2} \tag{A4}$$

The initial condition is obtained by substituting Eq. (18) into Eq. (11) to give

$$U(x,0) = \frac{\tau_0 - \tau_1}{\mu}(x - d) \tag{A5}$$

From Eqs.(13) and (18), boundary condition at *x* = *d* is

$$U(d,t) = 0 \tag{A6}$$

From Eqs. (14) and (A1), the boundary condition at *x* = 0 is

$$\frac{\partial U}{\partial x} = 0 \tag{A7}$$

The solution of U could be obtained as

$$U(x,t) = -\frac{2(\tau_1 - \tau_0)}{\mu d} \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} e^{-\nu \beta_n^2 t} \cos \beta_n x \tag{A8}$$

where

$$\beta_n = \frac{(2n+1)\pi}{2d} \ , n = 0,1,2 \ldots \tag{A9}$$

APPENDIX B

The initial condition is obtained by substituting Eq. (21) into Eq. (12) to give

$$U(x,0) = \frac{u_0 - u_1}{d}(d - x) \tag{B1}$$

From Eqs.(13) and (21), boundary condition at *x* = *d* is

$$U(d,t) = 0 \tag{B2}$$

From Eqs.(15) and (21), the boundary condition at *x* = 0 is

$$U(0,t) = 0 \tag{B3}$$

According to Eqs.(A2) to (A4) and Eqs.(B1) to (B3), U could be solved and expressed as

$$U(x,t) = -\frac{2(u_1 - u_0)}{d} \sum_{n=1}^{\infty} \frac{1}{\xi_n} e^{-\nu \xi_n^2 t} \sin \xi_n x \tag{B4}$$

where

$$\xi_n = \frac{n\pi}{d} \quad (n = 1,2,3 \ldots) \tag{B5}$$

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