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BERNOULLI'S TRANSFORMATION OF THE RESPONSE OF AN ELASTIC BODY AND DAMPING

Vasilios Nikitas[1], Alexandros Nikitas[2] and Nikolaos Nikitas[3*]

[1]Head of the Department of Building Structures
Prefecture of Drama
Drama 66100, Greece
E-mail: v-nikitas@otenet.gr

[2]Faculty of the Built Environment
University of the West of England
Frenchay Campus, Bristol BS16 1QY, UK
E-mail: Alexandros.Nikitas@uwe.ac.uk

[3]Department of Civil Engineering
University of Bristol
Queen's Building, University Walk, Bristol BS8 1TR, UK
E-mail: N.Nikitas@bristol.ac.uk

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ABSTRACT

Bernoulli's transformation and the related separation of variables method or modal analysis as classically applied to the partial differential equation of motion of an elastic continuum will always conclude an undamped response. However, this conclusion lacks reliability, since the underlying analysis assumes either integralwise differentiability (i.e. differentiation and integration signs are interchangeable) or termwise differentiability (i.e. the derivative of an infinite series of terms equals the sum of the derivatives of the terms) for Bernoulli's transformation, which not only is arbitrary but also is responsible for the undamped response.

This paper using Bernoulli's transformation examines an elastic uniform column ruled by the generalized Hooke's law and subjected to axial surface tractions at its free end or a free axial vibration, and shows that the above differentiability assumptions underlying classical analysis are equivalent and actually constitute a limitation to the class of the response functions. Only on this limitation, damping appears to be inconsistent with the elastic column response. Removing the limitation through nontermwise differentiability of Bernoulli's transformation results in a damped response of the elastic column, which indicates that damping actually complies with the generalized Hooke's law as applied to elastic continua.

1. INTRODUCTION

The paper is an advance on a publication of 1996 criticizing classical continuum dynamics as applied to a uniform elastic column subjected to an axial surface traction at its free end [1]. To specify, in the classical view, the elastic stresses do not include damping components, which by virtue of the boundary stress conditions deprives the surface traction of any damping components, thereby leading to an undamped response of the elastic column. In contrast, by means of Bernoulli's transformation [2 p.502-522], that is, using separation of variables method or modal analysis, this paper offers a mathematical proof that the elastic column response is damped, thereby verifying that the elastic stresses include damping components.

2. PARTIAL DIFFERENTIAL EQUATION OF MOTION OF A UNIFORM ELASTIC COLUMN

The continuum model of a uniform elastic column used in the above publication of 1996 [1] has been sketched in the following Fig. 1

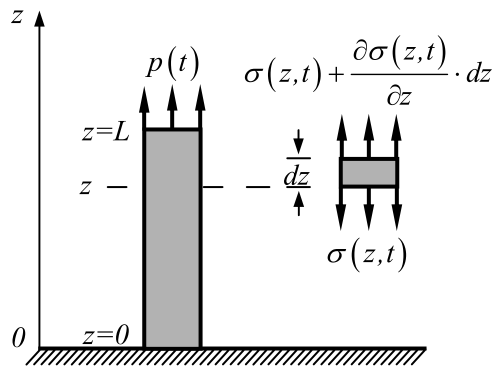

Figure 1. Uniform continuum model of an elastic column

The symbols used in Fig. 1 and also in the ensuing analysis are specified as below:

z stands for the level variable that refers to the initial state of static equilibrium of the elastic column. By definition, $z=0$ corresponds to the fixed end and $z=L$ corresponds to the free end of the column, with L standing for the initial (at-rest) length of the column.

t stands for the time variable.

$p(t)$ stands for an axial surface traction externally imposed on the column at its free end.

$\sigma(z,t)$ stands for the axial internal stress developed at the level z of the column.

The partial differential equation of motion of the uniform elastic column subjected to the surface traction $p(t)$ at its free end $z=L$ has the form

$$c^2 \cdot u''(z,t) - \ddot{u}(z,t) = 0\,, \tag{2.1}$$

where $u(z,t)$ and c stand for the displacement response of the cross-section level z of the elastic column at time t, and a constant, respectively. The constant c represents the velocity of propagation of longitudinal waves along the column and is equal to [3 p.408]

$$c = \sqrt{E/\rho}\,, \tag{2.2}$$

with E and ρ standing for the modulus of elasticity (Young's modulus) and the mass density of the uniform elastic column, respectively. Dots and primes over functions, e.g. $\ddot{u}(z,t)$ and $u''(z,t)$, stand for differentiation with reference to the time t and to the level z, respectively.

The partial differential equation (2.1) has been derived from Newton's second axiom as applied to the dynamics of an infinitesimal length of the elastic column at level z, viz.

$$\partial\sigma(z,t)/\partial z = \rho\cdot\ddot{u}(z,t), \tag{2.3}$$

and the generalized Hooke's law as a linear relation between stresses $\sigma(z,t)$ and strains $u'(z,t)$ at the same level [4 p.8], [5 p.58], that is,

$$\sigma(z,t) = E\cdot u'(z,t). \tag{2.4}$$

It is assumed that the elastic column continuum was at rest before subjected to the axial surface traction $p(t)$, and hence, its initial conditions $u(z,0)$ and $\dot{u}(z,0)$ must be zero,

$$u(z,0) = \dot{u}(z,0) = 0. \tag{2.5}$$

The influence of the axial surface traction $p(t)$ on the dynamics of the elastic column is accounted for through the inhomogeneous boundary stress condition

$$p(t) = \sigma(L,t) = E\cdot u'(L,t), \tag{2.6a}$$

while the fixed end at $z=0$ is represented by the homogeneous boundary condition

$$u(0,t) = 0. \tag{2.6b}$$

3. TRANSFORMATION OF AN INHOMOGENEOUS BOUNDARY VALUE INTO A HOMOGENEOUS ONE

The classical approach to the partial differential equation of motion of the elastic column of Fig. 1 requires homogeneous (i.e. zero) boundary conditions. Thus, a transformation of the response function in the partial differential equation (2.1) is sought so that the inhomogeneous boundary condition (2.6a) will be transformed into a homogeneous one [6 p.435-436]. Such a transformation of the response function $u(z,t)$ into a new one, say $\hat{u}(z,t)$, is given as below

$$\hat{u}(z,t) = u(z,t) - \frac{p(t)}{E}\cdot z. \tag{3.1}$$

On account of the above transformation, the partial differential equation (2.1), the initial conditions (2.5) and the boundary conditions (2.6), we conclude that the following partial differential equation in $\hat{u}(z,t)$ must hold true

$$c^2\cdot\frac{\partial^2\hat{u}(z,t)}{\partial z^2} - \frac{\partial^2\hat{u}(z,t)}{\partial t^2} = \frac{\ddot{p}(t)}{E}\cdot z, \tag{3.2}$$

with the homogeneous boundary conditions

$$\hat{u}(0,t) = \partial\hat{u}(L,t)/\partial z = 0\,, \tag{3.3}$$

and the initial conditions

$$\hat{u}(z,0) = -\frac{p(0)}{E}\cdot z \qquad \text{and} \qquad \frac{\partial\hat{u}(z,0)}{\partial t} = -\frac{\dot{p}(0)}{E}\cdot z\,. \tag{3.4}$$

4. UNCOUPLING THE PARTIAL DIFFERENTIAL EQUATION OF MOTION INTO UNDAMPED VIBRATIONS

The response $\hat{u}(z,t)$ of the column can be expressed as a Fourier series with respect to the system of orthogonal functions of modes $\varphi_n(z) = sin\left[(n+1/2)\pi z/L\right]$ as follows [7 p.489]

$$\hat{u}(z,t) \;=\; \sum_{n=0}^{\infty} q_n(t)\cdot\varphi_n(z) \;=\; \sum_{n=0}^{\infty} q_n(t)\cdot sin\left[(n+1/2)\pi z/L\right], \tag{4.1a}$$

$$q_n(t) \;=\; \left(\int_0^L \left[\varphi_n(z)\right]^2 dz\right)^{-1}\cdot\int_0^L u(z,t)\cdot\varphi_n(z)\cdot dz \;=\; \frac{2}{L}\int_0^L \hat{u}(z,t)\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz\,, \tag{4.1b}$$

with $q_n(t)$ standing for a Fourier coefficient. The modes $\varphi_n(z) = sin\left[(n+1/2)\pi z/L\right]$ are derived from the eigenvalue equation $\varphi_n''(z) + (\omega_n/c)^2\cdot\varphi_n(z) = 0$, with $\omega_n/c = constant$, and the homogeneous boundary conditions $\varphi_n(0) = \varphi_n'(L) = 0$ corresponding to the homogeneous boundary conditions (3.3). The Fourier series (4.1) is but Bernoulli's transformation of the response function $\hat{u}(z,t)$ [2 p.502-522], and by this latter name will be called in this paper.

By means of Bernoulli's transformation (4.1), classical analysis uncouples the partial differential equation of motion (3.2) into ordinary differential equations each one of which represents an undamped vibration in terms of a generalized displacement $q_n(t)$.

Indeed, multiplying the partial differential equation (3.2) by $sin\left[(n+1/2)\pi z/L\right]$ and integrating from $z=0$ to $z=L$ gives

$$c^2\cdot\int_0^L \frac{\partial^2\hat{u}(z,t)}{\partial z^2}\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz - \int_0^L \frac{\partial^2\hat{u}(z,t)}{\partial t^2}\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz =$$

$$= \frac{\ddot{p}(t)}{E}\cdot\int_0^L z\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz\,. \tag{4.2}$$

The first integral of the left-hand member after performing an integration by parts, and taking into account the homogeneous boundary conditions (3.3), becomes

$$\int_0^L \frac{\partial^2\hat{u}(z,t)}{\partial z^2}\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz = -\frac{(n+1/2)^2\pi^2}{L^2}\cdot\int_0^L \hat{u}(z,t)\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz\,, \tag{4.3}$$

which, after substituting $q_n(t)\cdot L/2$ for its right-hand member integral, in accordance with Bernoulli's transform (4.1b), may be rewritten as

$$\int_0^L \frac{\partial^2 \hat{u}(z,t)}{\partial z^2}\cdot sin\left[\left(n+1/2\right)\pi z/L\right]\cdot dz = -\left[\left(n+1/2\right)\pi/L\right]^2\cdot\frac{L}{2}\cdot q_n(t)\,. \tag{4.4}$$

The integral in the right-hand member of equation (4.2), in view of the following equality [8 p.164 eq. 216]

$$\int_0^L z\cdot sin\left[\left(n+1/2\right)\pi z/L\right]\cdot dz = \frac{(-1)^n}{\left[\left(n+1/2\right)\pi/L\right]^2}\,, \tag{4.5}$$

becomes equal to

$$\frac{\ddot{p}(t)}{E}\cdot\int_0^L z\cdot sin\left[\left(n+1/2\right)\pi z/L\right]\cdot dz = \frac{\ddot{p}(t)}{E}\cdot\frac{(-1)^n}{\left[\left(n+1/2\right)\pi/L\right]^2}\,. \tag{4.6}$$

Now, it is classically assumed that differentiating the integral in Bernoulli's transform (4.1b) to the second order with respect to time t equals differentiating its integrand to the second order with respect to time t (namely, *assumption of integralwise differentiability*), viz.

$$\ddot{q}_n(t) = \frac{2}{L}\int_0^L \frac{\partial^2 \hat{u}(z,t)}{\partial t^2}\cdot sin\left[\left(n+1/2\right)\pi z/L\right]\cdot dx\,, \tag{4.7}$$

for which, however, a sufficient condition is the continuity of the derivative $\partial^2\hat{u}(z,t)/\partial t^2$ for all values of z and t [7 p.286, Leibnitz's rule], [9 p.348]. And then, combining with equations (4.2), (4.4) and (4.6), the classical analysis can obtain the ordinary differential equations [10 p.212]

$$\ddot{q}_n(t) + \omega_n^2\cdot q_n(t) = -\frac{\ddot{p}(t)}{E}\frac{2L(-1)^n}{\left(n+1/2\right)^2\pi^2} \qquad \text{for} \quad n = 0,1,2,3,\dots, \tag{4.8}$$

where ω_n represents a natural angular frequency of the elastic column equal to

$$\omega_n = \left(n+1/2\right)\pi c/L\,. \tag{4.9}$$

Operating similarly, but without using any assumption, on the initial conditions (3.4) yields

$$q_n(0) = -\frac{p(0)}{E}\frac{2L(-1)^n}{\left(n+1/2\right)^2\pi^2} \qquad \text{and} \qquad \dot{q}_n(0) = -\frac{\dot{p}(0)}{E}\frac{2L(-1)^n}{\left(n+1/2\right)^2\pi^2}\,. \tag{4.10}$$

It is emphasized that each one of the ordinary differential equations (4.8) describes a forced vibration of a single-degree-of-freedom system in a generalized displacement $q_n(t)$, and by virtue of the initial conditions (4.10), is easy to be solved for a known surface traction $p(t)$. Moreover, the lack of a damping term in the left-hand member of each ordinary differential equation (4.8) suffices for the undamped nature of the forced vibration in $q_n(t)$.

Since the ordinary differential equations (4.8) are equivalent to the partial differential equation of motion (3.2) on the assumption of integralwise differentiability (4.7), their undamped character indicates the undamped character of the partial differential equation (3.2) on the assumption of integralwise differentiability (4.7). Only on this assumption, the classical view that the response of an elastic continuum is undamped seems to be reasonable.

5. A CRITIQUE OF THE UNCOUPLING INTO UNDAMPED VIBRATIONS

There are three major points of dispute over the uncoupling into the undamped vibrations (4.8) underlying the classical view that the elastic column response is undamped:

1. The undamped vibrations (4.8), and hence, the undamped character of the responses $\hat{u}(z,t)$ and $u(z,t)$, have been derived from uncoupling the partial differential equation of motion (3.2) exclusively on the assumption of integralwise differentiability (4.7).
2. The integralwise differentiability (4.7) can only be assured if the second-order derivative $\partial^2\hat{u}(z,t)/\partial t^2$ is continuous for all values of z and t, which, however, constitutes an arbitrary limitation to the response functions $\hat{u}(z,t)$ and $u(z,t)$, beyond the requirement of existence of the second-order derivative $\partial^2\hat{u}(z,t)/\partial t^2$ dictated by the partial differential equation (3.2). Thus, the assumption of integralwise differentiability (4.7) is but an arbitrary limitation to the elastic column continuum dynamics, which deprives the undamped vibrations (4.8) of any reliability as equivalent to the partial differential equation (3.2).
3. The assumption of integralwise differentiability (4.7) proves to be equivalent to the *assumption of termwise differentiability* of the infinite series (i.e. the derivative of the infinite series equals the sum of the derivatives of the series' terms) in Bernoulli's transform (4.1a). And hence, the really general solution to the elastic column continuum dynamics requires that the termwise differentiation rules be replaced by nontermwise differentiation rules.

The ensuing analysis elucidates these points and finally concludes that the really general solution to the elastic column continuum dynamics must represent a damped vibration.

6. EQUIVALENCE OF THE ASSUMPTIONS ABOUT INTEGRALWISE AND TERMWISE DIFFERENTIABILITY

As shown in par. 4, the ordinary differential equations (4.8) are equivalent to the partial differential equation (3.2) on the assumption of integralwise differentiability (4.7). With the aim to use this postulate, we can multiply equation (4.8) by $sin\left[(n+1/2)\pi z/L\right]$ and get

$$\ddot{q}_n(t)\cdot sin\left[(n+1/2)\pi z/L\right]+\omega_n^2\cdot q_n(t)\cdot sin\left[(n+1/2)\pi z/L\right]= \\ =-\frac{\ddot{p}(t)}{E}\frac{2L(-1)^n}{(n+1/2)^2\pi^2}\cdot sin\left[(n+1/2)\pi z/L\right], \tag{6.1}$$

which, taking into account that the second-order derivative of $sin\left[(n+1/2)\pi z/L\right]$ and equation (4.9) allow the substitution

$$\frac{d^2}{dz^2} sin\left[\left(n+1/2\right)\pi z/L\right] = -\left(\omega_n/c\right)^2 \cdot sin\left[\left(n+1/2\right)\pi z/L\right], \tag{6.2}$$

becomes

$$c^2 \cdot q_n\left(t\right) \cdot \frac{d^2}{dz^2} sin\left[\left(n+1/2\right)\pi z/L\right] - \ddot{q}_n\left(t\right) \cdot sin\left[\left(n+1/2\right)\pi z/L\right] =$$

$$= \frac{\ddot{p}\left(t\right)}{E} \frac{2L\left(-1\right)^n}{\left(n+1/2\right)^2 \pi^2} \cdot sin\left[\left(n+1/2\right)\pi z/L\right]. \tag{6.3}$$

Summing up all equations (6.3) for $n = 0,\ 1,2,3,\ldots$ entails

$$c^2 \cdot \sum_{n=0}^{\infty} q_n\left(t\right) \cdot \frac{d^2}{dz^2} sin\left[\left(n+1/2\right)\pi z/L\right] - \sum_{n=0}^{\infty} \ddot{q}_n\left(t\right) \cdot sin\left[\left(n+1/2\right)\pi z/L\right] =$$

$$= \frac{\ddot{p}\left(t\right)}{E} \cdot \frac{2L}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{\left(-1\right)^n}{\left(n+1/2\right)^2} \cdot sin\left[\left(n+1/2\right)\pi z/L\right]. \tag{6.4}$$

On account of Bernoulli’s transformation (4.1) and equation (4.5) it follows

$$z = \frac{2L}{\pi^2} \sum_{n=0}^{\infty} \frac{\left(-1\right)^n}{\left(n+1/2\right)^2} \cdot sin\left[\left(n+1/2\right)\pi z/L\right], \tag{6.5}$$

which inserting in equation (6.4) results in

$$c^2 \cdot \sum_{n=0}^{\infty} q_n\left(t\right) \cdot \frac{d^2}{dz^2} sin\left[\left(n+1/2\right)\pi z/L\right] - \sum_{n=0}^{\infty} \ddot{q}_n\left(t\right) \cdot sin\left[\left(n+1/2\right)\pi z/L\right] = \frac{\ddot{p}\left(t\right)}{E} \cdot z\,. \tag{6.6}$$

Equation (6.6) as a superposition of the ordinary differential equations (4.8) must be equivalent to the partial differential equation of motion (3.2), on the assumption of integralwise differentiability (4.7). This equivalence entails the *termwise differentiability* of Bernoulli’s transform (4.1a), viz.

$$\frac{\partial^2 \hat{u}\left(z,t\right)}{\partial t^2} = \sum_{n=0}^{\infty} \ddot{q}_n\left(t\right) \cdot sin\left[\left(n+1/2\right)\pi z/L\right]. \tag{6.7a}$$

$$\frac{\partial^2 \hat{u}\left(z,t\right)}{\partial z^2} = \sum_{n=0}^{\infty} q_n\left(t\right) \cdot \frac{d^2}{dz^2} sin\left[\left(n+1/2\right)\pi z/L\right]. \tag{6.7b}$$

Indeed, applying Bernoulli’s transformation (4.1) to the function $\partial^2 \hat{u}\left(z,t\right)/\partial t^2$, with this function replacing the function $\hat{u}\left(z,t\right)$ in the transformation, gives

$$\frac{\partial^2 \hat{u}\left(z,t\right)}{\partial t^2} = \sum_{n=0}^{\infty} g_n\left(t\right) \cdot sin\left[\left(n+1/2\right)\pi z/L\right], \tag{6.8a}$$

$$g_n\left(t\right)=\frac{2}{L}\int_0^L \frac{\partial^2 \hat{u}\left(z,t\right)}{\partial t^2}\cdot sin\left[\left(n+1/2\right)\pi z/L\right]\cdot dz\ , \tag{6.8b}$$

with $g_n\left(t\right)$ standing for a new Bernoulli's coefficient replacing $q_n\left(t\right)$. Further, on account of the assumption of integralwise differentiability (4.7), Bernoulli's transform (6.8b) yields

$$g_n\left(t\right)=\ddot{q}_n\left(t\right), \tag{6.9}$$

and substituting in Bernoulli's transform (6.8a) gives the termwise differentiation rule (6.7a), which combined with equation (6.6) and the partial differential equation of motion (3.2) results in the termwise differentiation rule (6.7b). And inversely, the assumption of termwise differentiability (6.7a) via Bernoulli's transformation (6.8) assures equality (6.9), and proves to be equal to the assumption of integralwise differentiability (4.7).

On this base therefore, Bernoulli's transformation (4.1) and the assumption of integralwise differentiability (4.7) entail the termwise differentiability (6.7). And inversely, Bernoulli's transformation (4.1) and the assumption of termwise differentiability (6.7) entail the integralwise differentiability (4.7), which implies that the assumptions of integralwise differentiability (4.7) and termwise differentiability (6.7) are equivalent. Thus, recalling that the assumption of integralwise differentiability (4.7) is the prerequisite for the undamped character of the partial differential equation (3.2), it follows that the undamped character of the partial differential equation (3.2) requires the termwise differentiability (6.7). That is, without the assumption of termwise differentiability (6.7), the partial differential equation (3.2) cannot describe any undamped motion, notwithstanding conventional wisdom.

7. UNCOUPLING THE PARTIAL DIFFERENTIAL EQUATION OF MOTION BY TERMWISE DIFFERENTIABILITY

The equivalence of the assumptions of integralwise differentiability (4.7) and termwise differentiability (6.7) can be verified by showing that the latter assumption in combination with the partial differential equation of motion (3.2) and Bernoulli's transformation (4.1) can result in the classical ordinary differential equations (4.8) like the former assumption.

Indeed, inserting the termwise differentiation rules (6.7) in the partial differential equation of motion (3.2), multiplying the resulting equation by $sin\left[\left(v+1/2\right)\pi z/L\right]$ for $v=1,2,3,\ldots$, integrating all over the column length L, and after having made use of the differentiation property (6.2) of the trigonometric function $sin\left[\left(n+1/2\right)\pi z/L\right]$ and its *orthogonality* property

$$\int_0^L sin\left[\left(n+1/2\right)\pi z/L\right]\cdot sin\left[\left(v+1/2\right)\pi z/L\right]\cdot dz=\begin{cases}0 & \text{for} \quad n\neq v\\ \dfrac{L}{2} & \text{for} \quad n=v\end{cases} \tag{7.1}$$

we can derive the classical ordinary differential equations (4.8) directly from the partial differential equation of motion (3.2) on the assumption of termwise differentiability (6.7).

8. INTEGRALWISE AND TERMWISE DIFFERENTIABILITY ASSUMPTIONS AS LIMITATIONS TO THE RESPONSE

The analysis in par. 4 made quite clear that the undamped ordinary differential equations (4.8) and their equivalence with the partial differential equation of motion (3.2), which suffices for the undamped character of the motion, is due to the arbitrary assumption of integralwise differentiability (4.7). Actually, Bernoulli's transform (4.1b) ensures this assumption only on the condition of a continuous derivative $\partial^2 \hat{u}(z,t)/\partial t^2$ for all values of z and t [7 p.286, Leibnitz's rule], [9 p.348], which, however, is beyond the requirement of the existence of the derivative $\partial^2 \hat{u}(z,t)/\partial t^2$ for the formation of the partial differential equation of motion (3.2). On the other hand, the derivative $\partial^2 \hat{u}(z,t)/\partial t^2$, as the acceleration of the column, is always discontinuous at the initial time $t=0$, where its at-rest zero value coexists with its nonzero value for leaving the state of rest. Besides, the theoretical possibility of abruptly applying an infinitesimal additional velocity distribution at a finite number of instants t in time, which, however, does not affect the forced vibration of the elastic column as a linear stable system, requires that the distribution of the acceleration $\partial^2 \hat{u}(z,t)/\partial t^2$ along z vary discontinuously with time at the finite number of instants t. In short, the continuity of the acceleration $\partial^2 \hat{u}(z,t)/\partial t^2$ sufficing for the integralwise differentiability (4.7), and hence, for the validity of Bernoulli's undamped ordinary differential equations (4.8), cannot be taken for consistent. Thus, the assumption of integralwise differentiability (4.7) is arbitrary and in fact constitutes a limitation to the class of the response functions $\hat{u}(z,t)$, thereby depriving the ordinary differential equations of the undamped vibrations (4.8) of any reliability as the real ordinary differential equations in the generalized displacements $q_n(t)$ that can be derived from the partial differential equation of motion (3.2). Exactly for this reason, the classical view that damping is inconsistent with the response of the elastic column, which is founded on the assumption of integralwise differentiability (4.7), is deprived of any reliability.

As its equivalence to the assumption of integralwise differentiability (4.7) indicates, the assumption of termwise differentiability (6.7) also is arbitrary [9 p.24-29] and in fact constitutes a limitation to the class of the response functions $\hat{u}(z,t)$. Actually, only the *uniform convergence* of the infinite series in equations (6.7) can suffice for the validity of the termwise differentiability of Bernoulli's transform (4.1a) [7 p.407-417], [9 p.24-30]. And of course, assuming this uniform convergence is but an arbitrary limitation to the class of the response functions $\hat{u}(z,t)$ sought.

9. THE DAMPING EFFECT OF NONTERMWISE DIFFERENTIABILITY

Either of the equivalent assumptions about integralwise differentiability (4.7) and termwise differentiability (6.7) actually constitutes a limitation to the class of the response functions $\hat{u}(z,t)$. So, in seeking the really general solution to the partial differential equation of motion (3.2), we can remove this limitation by replacing the termwise differentiation rules (6.7) with the *nontermwise* general differentiation rules

$$\frac{\partial^2 \hat{u}(z,t)}{\partial t^2} = \sum_{n=0}^{\infty} \ddot{q}_n(t) \cdot sin\left[(n+1/2)\pi z/L\right] + R_t(z,t), \tag{9.1a}$$

$$\frac{\partial^2 \hat{u}(z,t)}{\partial z^2} = \sum_{n=0}^{\infty} q_n(t) \cdot \frac{d^2}{dz^2} sin\left[(n+1/2)\pi z/L\right] + R_z(z,t), \tag{9.1b}$$

where $R_t(z,t)$ and $R_z(z,t)$ stand for *remainder terms* that cannot be separated into the modal forms included in the series under the summation sign .

Inserting equations (9.1) in the partial differential equation of motion (3.2) will give

$$c^2 \cdot \sum_{n=0}^{\infty} q_n(t) \cdot \frac{d^2}{dz^2} sin\left[(n+1/2)\pi z/L\right] - \sum_{n=0}^{\infty} \ddot{q}_n(t) \cdot sin\left[(n+1/2)\pi z/L\right] +$$

$$+c^2 \cdot R_z(z,t) - R_t(z,t) = \frac{\ddot{p}(t)}{E} \cdot \frac{2L}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1/2)^2} \cdot sin\left[(n+1/2)\pi z/L\right], \tag{9.2}$$

which after using equation (6.2) and rearranging terms becomes

$$\sum_{n=0}^{\infty} \left\{ \ddot{q}_n(t) + \omega_n^2 \cdot q_n(t) \right\} \cdot sin\left[(n+1/2)\pi z/L\right] + R_t(z,t) - c^2 \cdot R_z(z,t) =$$

$$= -\frac{\ddot{p}(t)}{E} \cdot \frac{2L}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1/2)^2} \cdot sin\left[(n+1/2)\pi z/L\right]. \tag{9.3}$$

On account of equation (9.3), we can always express the sum $R_t(z,t) - c^2 \cdot R_z(z,t)$ by means of Bernoulli's transformation (4.1) as follows

$$R_t(z,t) - c^2 \cdot R_z(z,t) = \sum_{n=0}^{\infty} \omega_n^2 \cdot y_n(t) \cdot sin\left[(n+1/2)\pi z/L\right], \tag{9.4a}$$

$$\omega_n^2 \cdot y_n(t) = \frac{2}{L} \int_0^L \left\{ R_t(z,t) - c^2 \cdot R_z(z,t) \right\} \cdot sin\left[(n+1/2)\pi z/L\right] \cdot dz, \tag{9.4b}$$

where $\omega_n^2 \cdot y_n(t)$ stands for a generalized acceleration, and hence, $y_n(t)$ denotes a new generalized displacement in addition to the generalized displacement $q_n(t)$. Notwithstanding that the generalized acceleration $\omega_n^2 \cdot y_n(t)$ appears as a product, it actually represents a single Bernoulli's coefficient only, say, $\omega_n^2 \cdot y_n(t) = \psi_n(t)$.

Inserting Bernoulli's transform (9.4a) in equation (9.3) yields

$$\sum_{n=0}^{\infty} \left\{ \ddot{q}_n(t) + \omega_n^2 \cdot \left[q_n(t) + y_n(t) \right] \right\} \cdot sin\left[(n+1/2)\pi z/L\right] =$$

$$= -\frac{\ddot{p}(t)}{E} \cdot \frac{2L}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1/2)^2} \cdot sin\left[(n+1/2)\pi z/L\right]. \tag{9.5}$$

Multiplying equation (9.5) by $sin\left[(v+1/2)\pi z/L\right]$ with $v = 1,2,3,\ldots$, integrating the resulting equation all over the length L of the column, interchanging the order of integration and summation, and taking into account the orthogonality property (7.1), we can finally conclude that equation (9.5) is equivalent to

$$\ddot{q}_n(t) + \omega_n^2 \cdot \left[q_n(t) + y_n(t) \right] = -\frac{\ddot{p}(t)}{E} \frac{2L(-1)^n}{(n+1/2)^2 \pi^2} \qquad \text{for} \quad n = 0,1,2,3,\ldots \tag{9.6}$$

In view of equation (4.8) and the analysis of par. 4, the generalized displacement $y_n(t)$ can only be zero on the assumption of the integralwise differentiability (4.7), that is,

$$y_n(t)=0 \quad \Leftrightarrow \quad \textit{assumption of integralwise differentiability (4.7)}. \tag{9.7}$$

Equations (9.6) are the actual ordinary differential equations in the generalized displacements $q_n(t)$ that can in general be derived from the partial differential equation of motion (3.2). Since the ordinary differential equations (9.6) are equivalent to equation (9.5), which actually represents the partial differential equation of motion (3.2), they must be equivalent to the partial differential equation of motion (3.2). And hence, the undamped or damped character of the ordinary differential equations (9.6) indicates the undamped or damped, respectively, character of the partial differential equation of motion (3.2). Let us examine whether the actual ordinary differential equations (9.6) represent undamped or damped vibrations.

Evidently, each ordinary differential equation (9.6) describes a forced vibration of a single-degree-of-freedom system in a generalized displacement $q_n(t)$, with the system internal force $\omega_n^2 \cdot \left[q_n(t)+y_n(t)\right]$ having magnitude dependent not only on the displacement $q_n(t)$ of its application point, which implies that its work does not depend only on the displacement $q_n(t)$. This latter assures that the internal force $\omega_n^2 \cdot \left[q_n(t)+y_n(t)\right]$ must be nonconservative [11 p.90], [12 p.3-4], which verifies that the actual ordinary differential equation (9.6) must represent a damped vibration of the single-degree-of-freedom system.

On this base therefore, the partial differential equation (3.2) must in general represent a damped vibration. Only assuming the integralwise differentiability (4.7) of Bernoulli's transform (4.1b), which implies $y_n(t)=0$, the internal force $\omega_n^2 \cdot \left[q_n(t)+y_n(t)\right]$ becomes equal to $\omega_n^2 \cdot q_n(t)$, and hence, conservative, thereby reducing the actual ordinary differential equation (9.6) to the classical ordinary differential equation (4.8), which represents an undamped vibration.

10. THE REALLY GENERAL SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION OF MOTION

The velocity magnitude $\omega_n \cdot y_n(t)$ can be related to the generalized velocity $\dot{q}_n(t)$ as below

$$\omega_n \cdot y_n(t)=2\xi_n(t)\cdot \dot{q}_n(t), \tag{10.1}$$

where $\xi_n(t)$ denotes a scalar coefficient varying with time so that it ensures equation (10.1). Actually, $\xi_n(t)$ is a function of both the generalized magnitudes $y_n(t)$ and $\dot{q}_n(t)$.

Applying substitution (10.1) to the actual ordinary differential equation (9.6) yields

$$\ddot{q}_n(t)+2\xi_n(t)\cdot\omega_n\cdot\dot{q}_n(t)+\omega_n^2\cdot q_n(t)=-\frac{\ddot{p}(t)}{E}\frac{2L(-1)^n}{(n+1/2)^2\,\pi^2} \qquad \text{for} \quad n=0,1,2,3,\dots, \tag{10.2}$$

which discloses that the scalar coefficient $\xi_n(t)$ is but the *damping ratio* of the n^{th} mode. Thus, the partial differential equation (3.2) is actually uncoupled into the nonlinear ordinary differential equations (10.2), which represent damped vibrations, and not into the classical ordinary differential equations (4.8), which represent undamped vibrations.

Inserting the damped general solution of the actual nonlinear ordinary differential equation (10.2) in Bernoulli's transformation (4.1) and combining with equation (3.1) gives the really general solution of the initial-boundary value problem of the uniform elastic column, which must also be damped in view of the equivalence of the differential equations (10.2) and (3.2).

In the case of an almost constant damping ratio ξ_n or a mean damping ratio ξ_n, the actual differential equation (10.2) may be simplified to [13 p.12-15, p.48]

$$\ddot{q}_n(t) + 2\xi_n \cdot \omega_n \cdot \dot{q}_n(t) + \omega_n^2 \cdot q_n(t) = -\frac{\ddot{p}(t)}{E}\frac{2L(-1)^n}{(n+1/2)^2\,\pi^2} \qquad \text{for} \quad n = 0,1,2,3,\dots, \tag{10.3}$$

whose general solution, for the *underdamped* case $0 < \xi_n < 1$, equals [3 p.234 eq.(5.22)]

$$\begin{aligned} q_n(t) = {} & e^{-\xi_n\cdot\omega_n\cdot t}\cdot\left\{q_n(0)\cdot\cos\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right] + \frac{\dot{q}_n(0)+\xi_n\cdot\omega_n\cdot q_n(0)}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\sin\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]\right\} + \\ & -\frac{1}{\sqrt{1-\xi_n^2}\cdot\omega_n}\frac{2L(-1)^n}{(n+1/2)^2\,\pi^2}\frac{1}{E}\cdot\int_0^t \ddot{p}(\tau)\cdot e^{-\xi_n\cdot\omega_n\cdot(t-\tau)}\cdot\sin\left[\omega_n(t-\tau)\right]\cdot d\tau\,. \end{aligned} \tag{10.4}$$

On account of the classical initial conditions (4.10), which are independent of the assumption of integralwise or termwise differentiability, the general solution (10.4) becomes

$$\begin{aligned} q_n(t) = \frac{2L(-1)^n}{(n+1/2)^2\,\pi^2}\frac{1}{E}\cdot\Bigg\{ & e^{-\xi_n\cdot\omega_n\cdot t}\cdot\Bigg(p(0)\cdot\cos\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right] + \\ & + \frac{\dot{p}(0)+\xi_n\cdot\omega_n\cdot p(0)}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\sin\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]\Bigg) + \\ & + \frac{1}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\int_0^t \ddot{p}(\tau)\cdot e^{-\xi_n\cdot\omega_n\cdot(t-\tau)}\cdot\sin\left[\omega_n(t-\tau)\right]\cdot d\tau\Bigg\}, \end{aligned} \tag{10.5}$$

and substituting in Bernoulli's transform (4.1a) and using equation (4.9) gives

$$\begin{aligned} \hat{u}(z,t) = \frac{2c^2}{L\cdot E}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{\omega_n^2}\cdot\Bigg\{ & e^{-\xi_n\cdot\omega_n\cdot t}\cdot\Bigg(p(0)\cdot\cos\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right] + \\ & + \frac{\dot{p}(0)+\xi_n\cdot\omega_n\cdot p(0)}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\sin\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]\Bigg) + \\ & + \frac{1}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\int_0^t \ddot{p}(\tau)\cdot e^{-\xi_n\cdot\omega_n\cdot(t-\tau)}\cdot\sin\left[\omega_n(t-\tau)\right]\cdot d\tau\Bigg\}\cdot\sin\frac{\omega_n}{c}z\,, \end{aligned} \tag{10.6}$$

which is the really general solution of the partial differential equation (3.2).

Combining equations (10.6), (3.1), (4.9) and (6.5), we conclude the really general solution of the partial differential equation of motion (2.1) for $\xi_n(t)=\xi_n=constant$, as below

$$u(z,t)=\frac{2c^2}{L\cdot E}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{\omega_n^2}\cdot\left\{p(t)-e^{-\xi_n\cdot\omega_n\cdot t}\cdot\left(p(0)\cdot cos\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]+\right.\right.$$

$$\left.+\frac{\dot{p}(0)+\xi_n\cdot\omega_n\cdot p(0)}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot sin\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]\right)+$$

$$\left.-\frac{1}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\int_0^t\ddot{p}(\tau)\cdot e^{-\xi_n\cdot\omega_n\cdot(t-\tau)}\cdot sin\left[\omega_n(t-\tau)\right]\cdot d\tau\right\}\cdot sin\frac{\omega_n}{c}z\,, \tag{10.7}$$

which represents a superposition of damped forced vibrations of the elastic column, thereby representing a damped response of the elastic column.

11. THE CASE OF FREE AXIAL VIBRATION

A free axial vibration of the elastic column corresponds to zero surface traction, i.e. $p(t)=0$, and hence, to $\dot{p}(t)=\ddot{p}(t)=0$, which implies that $\hat{u}(z,t)=u(z,t)$, and is excited by the general initial conditions $u(z,0)$ and $\dot{u}(z,0)$. In this case, the partial differential equation of motion remains the same as equation (2.1), while Bernoulli's transformation (4.1) and the continuity of $u(z,t)$ and $\dot{u}(z,t)$, which is assured by the existence of the second-order derivatives $\partial^2\hat{u}(z,t)/\partial t^2$ and $\partial^2\hat{u}(z,t)/\partial z^2$, lead to the following initial conditions for $q_n(t)$

$$q_n(0)=\frac{2}{L}\int_0^L u(z,0)\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz\,, \tag{11.1a}$$

$$\dot{q}_n(0)=\frac{2}{L}\int_0^L \dot{u}(z,0)\cdot sin\left[(n+1/2)\pi z/L\right]\cdot dz\,. \tag{11.1b}$$

All preceding equations also hold true for a free axial vibration of the elastic column by inserting the above general conditions. For example, the general solution (10.4) after putting $\ddot{p}(t)=0$ and substituting $q_n(0)$ and $\dot{q}_n(0)$ in accordance with the initial conditions (11.1) becomes the general solution in $q_n(t)$ of the free axial vibration. This latter solution inserting in Bernoulli's transform (4.1a) and after replacing $\hat{u}(z,t)$ by $u(z,t)$ gives the really general solution of the partial differential equation (2.1) for $\xi_n(t)=\xi_n=constant$, viz

$$u(z,t)=\frac{2}{L}\cdot\sum_{n=0}^{\infty}e^{-\xi_n\cdot\omega_n\cdot t}\cdot sin\frac{\omega_n}{c}z\cdot\left\{\left(\int_0^L u(z,0)\cdot sin\frac{\omega_n}{c}z\cdot dz\right)\cdot cos\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]+\right.$$

$$\left.+\frac{1}{\sqrt{1-\xi_n^2}\cdot\omega_n}\cdot\left(\int_0^L\left[\dot{u}(z,0)+\xi_n\cdot\omega_n\cdot u(z,0)\right]\cdot sin\frac{\omega_n}{c}z\cdot dz\right)\cdot sin\left[\sqrt{1-\xi_n^2}\cdot\omega_n t\right]\right\}, \tag{11.2}$$

which represents a superposition of damped free axial vibrations of the elastic column, thereby representing a damped response of the elastic column.

12. REMARK

It is worth emphasizing that the really general solutions (10.7) and (11.2) have been derived on the assumption of constant damping ratios ξ_n, which limits the actual ordinary differential equations (10.2) to linear equations only. For the general case of varying damping ratios $\xi_n(t)$, equations (10.2) become nonlinear, thereby representing nonlinear damped vibrations in the generalized displacements $q_n(t)$, and hence, the really general solution of the partial differential equation (2.1) becomes a superposition of these nonlinear damped vibrations.
Only on the arbitrary assumption of integralwise differentiability (4.7) or its equivalent assumption of termwise differentiability (6.7), equations (9.7) and (10.1) make the damping ratio ξ_n be zero, thereby reducing each of the really general solutions (10.7) and (11.2) to a superposition of undamped vibrations, which represents the classical case.

13. CONCLUSIONS

On the base of the above analysis, our criticism of the sufficiency of the classical application of Bernoulli's transformation to the dynamics of an elastic continuum seems to be justified. Actually, by Bernoulli's transformation itself we conclude that the really general solution of the partial differential equation of motion of an elastic column subjected to a surface traction at its free end or an initial excitation represents a superposition of damped vibrations, thereby representing a damped response. The undamped response of classical analysis results from arbitrary assumptions about the differentiability of Bernoulli's transformation that reduce the really general solution to a superposition of undamped vibrations.
The damped response indicates that the stresses developed in the elastic column must include damping components, thereby indicating that damping actually complies with the generalized Hooke's law as applied to elastic continua.

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