NONCONSERVATIVE NATURE OF THE STRESSES DEVELOPED IN A CONTINUUM

Vasilios Nikitas[1], Alexandros Nikitas[2] and Nikolaos Nikitas[3*]

[1] Head of the Directorate of Development

Prefecture of Drama

Drama 66100, Greece

E-mail: v-nikitas@otenet.gr

[2] School of the Built and Natural Environment

University of the West of England

Frenchay Campus, Bristol BS16 1QY, UK

E-mail: Alexandros.Nikitas@uwe.ac.uk

[3] Department of Civil Engineering

University of Bristol

Queen's Building, University Walk, Bristol BS8 1TR, UK

E-mail: N.Nikitas@bristol.ac.uk

[*] Corresponding author

Abstract

The stresses in an elastic continuum (i.e. a continuum with zero strains after unloading) are classically deemed to be conservative (i.e. their total work all over the continuum is a single-valued function of only the displacement distribution in the continuum). So, internal damping in an elastic continuum appears to be a contradiction in itself.

Actually, the total work of the internal stresses all over a continuum does not coincide with the strain energy of the continuum, but also includes the work of the internal body forces formed by the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\dots$, which only contributes to the kinetic energy of the continuum. Owing to this inclusion, the total work of the internal stresses cannot be a single-valued function of only the displacement distribution in the continuum, and hence, the internal stresses must be nonconservative, which indicates internal damping inherent in any continuum whether elastic or not. Only statically deforming continua may possess conservative internal stresses.

Keywords: Elasticity and damping, nonconservative nature of internal stresses.

Contents

Nomenclature

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	shearing strains
Δ	prefix standing for 'additional'
$\Delta X, \Delta Y, \Delta Z$	additional external body forces
$\Delta T_x, \Delta T_y, \Delta T_z$	additional surface tractions
$\Delta(\partial\sigma_{xx}/\partial x)$	additional stress derivative
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$	normal strains
$\boldsymbol{\varepsilon}(x,y,z;t)$	column matrix of strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$
$\mathbf{F}$	column matrix of the internal forces of a system of mass points, column matrix of the forces resulting from the internal stresses acting on the sides of an infinitesimal parallelepiped
$\mathbf{F}_i$	resultant internal force in vector form at the i mass point
$\Theta(t)$	internal energy of a continuum
$\boldsymbol{\kappa}$	square matrix of elastic constants
$Q(t)$	heat supplied to a continuum
ρ	mass density
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	normal internal stresses
$\boldsymbol{\sigma}(x,y,z;t)$	stress tensor at the point (x,y,z) of a continuum as the column matrix of the balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ acting on the sides of an infinitesimal parallelepiped enclosing the point (x,y,z)
t	time
T_x, T_y, T_z	surface tractions

$\mathbf{T}(x,y,z;t)$	column matrix of surface tractions at point (x,y,z)
$\tau, \hat{\tau}$	integrand variable for time t
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	shearing internal stresses
$\mathbf{U}$	displacement matrix of a system, column matrix of the displacements of the sides of an infinitesimal parallelepiped due to only the deformation of the parallelepiped
$\hat{\mathbf{U}}$	integrand variable for the displacement matrix $\mathbf{U}$
u_x, u_y, u_z	displacements at a point of a continuum
$\mathbf{u}_i$	displacement vector of the i mass point from its equilibrium position
$\hat{\mathbf{u}}_i$	integrand variable for the displacement vector $\mathbf{u}_i$
W	total work of the internal forces of a system of interacting mass points
W_i	work of the internal force $\mathbf{F}_i$
X, Y, Z	external body forces per unit of volume
$\mathbf{X}(x,y,z;t)$	column matrix of external body forces X, Y, Z at point (x,y,z)

1. Consistency of elasticity with damping

The demarcation of science from metaphysics can be crystallized as follows [1 p.1]:
"Every scientific theory starts from a set of hypotheses, which are suggested by our observations, but represent an idealization of them. The theory is then tested by checking the predictions deduced from these hypotheses against experiment. When persistent discrepancies are found, we try to modify the hypotheses to restore the agreement with observation. If many such tests are made and no serious disagreement emerges, then the hypotheses gradually acquire the status of 'laws of nature'. When results that apparently contradict well-established laws appear, as they often do, we tend to look for other possible explanations-for simplifying approximations we have made that may be wrong, or neglected effects that may be significant."
On this base therefore, the discrepancy between the classical hypothesis that "elastic means an absence of damping forces" [2] and the observation of damping and hysteresis loops in elastic bodies [3 p.120] calls for a reasonable explanation. That is, we need to carefully investigate and re-examine whether or not the real nature of the stresses developed in the continuum model of the elastic bodies is conservative.

As is well known, *elasticity* consists in a force-deformation or stress-strain relation, which allows a structure to recover its initial unstrained configuration, thereby excluding any residual strains, after removing the applied loads [3 p.92], [4 p.1]. In Sokolnikoff's words: "A body is called *elastic* if it possesses the property of recovering its original shape when the forces causing deformations are removed. … The elastic property is characterized mathematically by certain functional relationships connecting forces and deformations." [4 p.1]. A body is therefore *inelastic* if it exhibits residual deformation (residual strains) after loading removal.

Bodies are modelled as continuous or discrete systems. In both models, elasticity, by its very definition, does not impose any absence of damping forces, notwithstanding the classical hypothesis that "elastic means an absence of damping forces" [2]. After all, if elastic meant absence of damping forces, then abruptly removing all loads would lead an elastic system to an everlasting free undamped vibration, which cannot comply with the classical definition of elasticity mentioned above (i.e. no strains after unloading). Besides, a lot of engineering structures (e.g. buildings or bridges subject to earthquakes or winds, aircrafts subject to air flow) can undergo damped vibrations with no residual strains after unloading. Such a structural behaviour rather indicates that damping complies with elasticity and 'elastic' may be 'damped' as well as 'inelastic' is.

On the other hand, the hypothesis that 'elastic' means 'undamped' [2] not only is arbitrary but also requires that no damping surface tractions be applied to an elastic body. Indeed, owing to the classical boundary stress conditions [5 pp.28-29,236], the internal elastic stresses at a point of the boundary surface of an elastic continuum must equal the surface tractions applied at this point. So, if 'elastic' meant 'undamped', then the internal elastic stresses could not include damping components, and hence, the surface tractions could not include damping components either. That is, in the classical view, elasticity cannot comply with nonconservative surface tractions, which implies that these two entities cannot coexist in nature. However, in nature, we can realize nonconservative surface tractions (e.g. velocity-dependent wind surface tractions) applied to elastic bodies (i.e. bodies recovering its original shape after unloading), which indicates that *nonconservative stresses, and hence, damping, can comply with elasticity* despite the classical hypothesis that 'elastic' means 'undamped'.

2. Classical model of internal stresses

We shall first review the classical model of internal stresses in a continuum and specify what the internal stresses and their spatial derivatives, the total work of the internal stresses, the strain energy and the kinetic energy are and how they are interrelated.

Let us consider a deformable body as a continuum subjected to dynamical deformation by externally applied loads. Owing to the deformation of the whole body, an infinitesimal element of the body deforms and also moves as if it were a rigid particle. Thus, the energy supplied by the deformation of the whole body to the infinitesimal element consists of two parts: the strain energy due to the deformation of the element itself and the kinetic energy due to the motion of the element as a rigid particle [6 p.261].

We now focus on the infinitesimal element, which is sketched in Fig. 1 as an orthogonal parallelepiped for an interior element, or an orthogonal triangular pyramid with base on the boundary surface of the body for a boundary element, with dimensions $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ [7 pp.20-21]. Its interactions with the adjacent infinitesimal elements as well as the surface loading externally applied to it in the case of a boundary element may be represented as stresses acting on the boundary surface of the element, which are herein called *internal stresses*. The internal stresses act in addition to possible body forces externally applied to the volume of the element. The total work performed by the internal stresses acting on the infinitesimal element during the transition from the unstrained configuration (i.e. the natural configuration with no deformation) of the body to its strained (deformed) configuration consists of two parts: The work of the *balanced* components of the internal stresses and the work of the *unbalanced* components of the internal stresses.

The balanced components of the internal stresses of the infinitesimal element, that is, the classical stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ acting at the element's three pairs of oppo

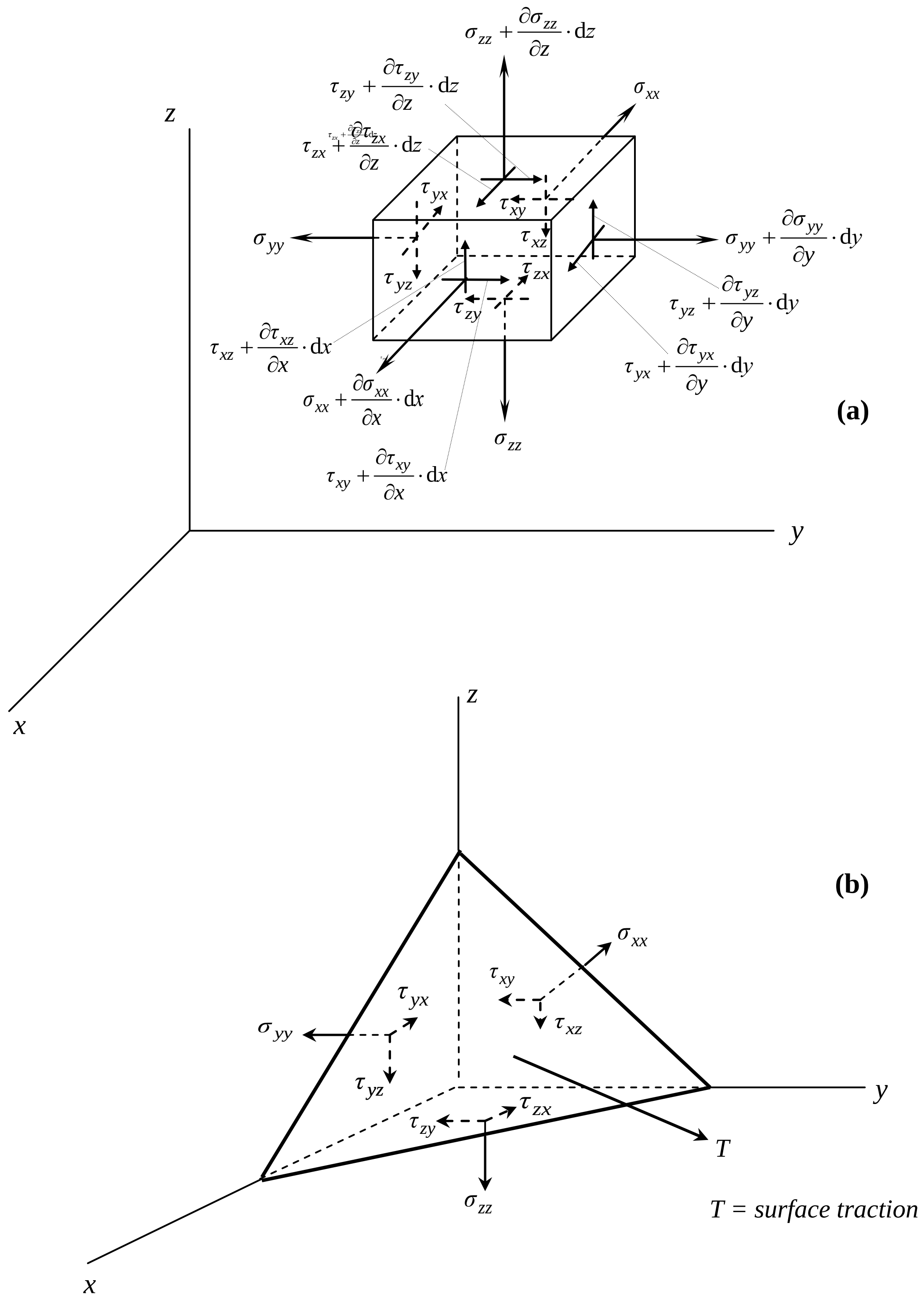

Fig. 1: (a) The total stresses on the boundary surface of an infinitesimal element.

(b) The total stresses of an infinitesimal element on the boundary surface of a continuous structure.

site sides with a zero resultant force, are exclusively responsible for the deformation of the infinitesimal element, and their work equals the *strain energy* of the element.

The unbalanced components of the internal stresses of the infinitesimal element, that is, the internal-stress differences $(\partial\sigma_{xx}/\partial x)\mathrm{d}x,\ (\partial\tau_{yx}/\partial y)\mathrm{d}y,\ (\partial\tau_{zx}/\partial z)\mathrm{d}z,\ldots$, result in the action of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ as internal body forces, which together with the externally applied body forces are exclusively responsible for the motion of the element as described by the classical differential equations [3 p.85 eq.(15)]

$$\left.\begin{aligned}
&\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=\rho\cdot\frac{\partial^2 u_x}{\partial t^2}\\
&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=\rho\cdot\frac{\partial^2 u_y}{\partial t^2}\\
&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+Z=\rho\cdot\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}\right\}\qquad(1)$$

where $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ are internal body forces per unit of volume resulting from the unbalanced stress scomponents $(\partial\sigma_{xx}/\partial x)\mathrm{d}x,\ (\partial\tau_{yx}/\partial y)\mathrm{d}y,\ (\partial\tau_{zx}/\partial z)\mathrm{d}z,\ldots$. These latter compared with the balanced stress components $\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{yz},\tau_{xz}$ are negligible with no effect on strains, although they produce finite motion.

X,Y,Z stand for the components of the external body forces per unit of volume of the infinitesimal element along the x,y,z coordinate axes, respectively.

u_x,u_y,u_z stand for the displacements of the infinitesimal element along the x,y,z coordinate axes, respectively.

ρ stands for the mass density of the infinitesimal element.

From the above differential equations it follows [3 p.94] that the sum of the work done by the internal body forces $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\dots$ and the work done by the external body forces X,Y,Z equals the *kinetic energy* of the infinitesimal element. As a consequence, the work done by the internal body forces (i.e. the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\dots$) acting on an infinitesimal element exclusively contributes to the formation of kinetic energy for the infinitesimal element.

Within this frame, for either an infinitesimal element of a body or the entire body, the total work of the internal stresses, the strain energy and the work done by the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\dots$ are interrelated as follows

$$\textit{Total work of internal stresses} = \textit{strain energy} + \textit{work of stress derivatives}. \qquad (2)$$

It is emphasized that in many vibration problems only surface loads act on the structures. In this case, equations (1) imply that the work done by the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\dots$ coincides with the kinetic energy of the body.

3. The notion of conservative forces and stresses

Actually, a stress is but a surface force per unit area [3 p.74], [4 p.36], [8 p.47], [9 pp.4-8]. So, a stress at a point (i.e. at a side of an infinitesimal parallelepiped) is *conservative* as long as the surface force that results from the stress is conservative. In the classical theory [10 pp.3-4], [11 pp.390-391,418], [12 pp.347,360-361], [13 pp.90-91], [14 p.247], an individual force is classified as conservative, if its work done along any path of its application point is a *single-valued* function of *only* the positions of the end points of the path, thereby being independent of the path and zero along any closed path.

Let us now consider a system of interacting mass points whose configuration deforms under a loading. The deformation of the configuration consists in different displacements of the mass points around their equilibrium positions (i.e. the positions defining the unstrained configuration), and can completely be described by the displacement vectors of the mass points relative to their equilibrium positions [13 pp.162-163], [15 p.11,29]. As an effect of the deformation, internal forces develop within the system. In line with the mentioned classical definition of a conservative force, the resultant $\mathbf{F}_i$ of the internal forces acting on the i mass point of the system is classified as conservative, if its work W_i done when the i mass point moves from its equilibrium position with zero displacement vector up to a position with displacement vector $\mathbf{u}_i$ is a single-valued function of only the displacement vector $\mathbf{u}_i$, say $W_i(\mathbf{u}_i)$. This classical definition is mathematically expressed as [1 pp.14-15 eq.(2-5)], [8 p.3], [13 p.91 eq.(8-8)]

$$\int_0^{\mathbf{u}_i} \mathbf{F}_i \cdot \mathrm{d}\hat{\mathbf{u}}_i = W_i(\mathbf{u}_i), \qquad \textit{for a conservative internal force } \mathbf{F}_i, \tag{3}$$

where $\hat{\mathbf{u}}_i$ is the integrand variable for the displacement vector $\mathbf{u}_i$ of the i mass point.

Similarly, the total of the internal forces $\mathbf{F}_i$ of the system, and hence, their column matrix $\mathbf{F}$, can be classified as conservative, if the sum W of the works of all internal forces $\mathbf{F}_i$ done along the displacement matrix $\mathbf{U}$ of the system (i.e. the displacements of the mass points of the system) is a single valued function of only the displacement matrix

$\mathbf{U}$ [10 p.10], [13 pp.92-93,162]. This classical definition is mathematically expressed as follows

$$\int_0^{\mathbf{U}} \mathbf{F}^T \cdot \mathrm{d}\hat{\mathbf{U}} = W(\mathbf{U}), \qquad \textit{for a conservative column matrix } \mathbf{F} \textit{ of internal forces}, \qquad (4)$$

where $\mathbf{F}$ is the column matrix of the internal forces acting on the mass points,

$\mathbf{U}$ is the column matrix of the displacements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ of the mass points,

$\hat{\mathbf{U}}$ is the integrand variable for the displacement vector $\mathbf{U}$,

$W(\mathbf{U})$ represents the sum of the works of the internal forces $\mathbf{F}_i$ as a single-valued function of only the displacement matrix $\mathbf{U}$,

T as an upper index denotes the transpose of the matrix indexed.

Within the frame of classical theory, by analogy with the conservative total of the internal forces [13 p.93], the total of the internal stresses in a continuum can be classified as conservative, if and only if their total work is a single-valued function of only the displacement distribution in the continuum. This latter condition means that the balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ at (x,y,z), which define the stress tensor $\boldsymbol{\sigma}(x,y,z;t)$, and the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$, which result from the unbalanced stress components $(\partial\sigma_{xx}/\partial x)\mathrm{d}x,\ (\partial\tau_{yx}/\partial y)\mathrm{d}y,\ (\partial\tau_{zx}/\partial z)\mathrm{d}z, \ldots$ and define the internal body forces at (x,y,z), perform works whose sum all over the continuum is a single-valued function of only the displacement distribution in the continuum.

4. Equality of the total work of internal stresses with the work of surface tractions

By Newton's third axiom of action and reaction, the internal stresses of two adjacent infinitesimal elements acting on their common boundary are equal with opposite directions, thereby performing zero total work. And hence, the total work of the internal stresses all over the body equals the total work of the internal stresses acting on the body's boundary surface, which is not common boundary of adjacent elements [6 pp.261-262]. Thus, by virtue of the classical boundary stress conditions, i.e. equality of the internal stresses on a body's boundary surface with the surface tractions (that is to say, the external stresses) [5 pp.28-29,236], *the total work of the internal stresses all over a body must equal the work of the surface tractions*.

On this base therefore, the internal stresses developed all over a continuum can be classified as conservative if and only if the work of the surface tractions is a single-valued function of only the displacement distribution in the continuum.

5. Mathematical proof of the nonconservative nature of the internal stresses

As exposed in par. 3, for conservative internal stresses in a continuum, the sum of the works performed by the balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ and the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$ all over the continuum should be a single valued function of only the displacement distribution in the continuum.

However, the above single-valuedness cannot actually be satisfied, which proves that the internal stresses are nonconservative, thereby including damping components.

Indeed, according to the classical model of internal stresses exposed in par. 2, the displacement at each point of a continuum depends only on the history of the external body forces X, Y, Z and the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \dots$ acting at the point and governing the motion of the point via equations (1), and not on the balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ at the point. After all, the balanced stresses acting on the boundary of the infinitesimal element surrounding the point build up a zero resultant force, which cannot influence the motion (and hence, the displacement) of the point.

Let us now consider a continuum subjected to a given history of surface tractions T_x, T_y, T_z and external body forces X, Y, Z, which, owing to the linearity of the differential equations of motion (1) and the boundary stress conditions, implies a unique history of internal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ [7 pp.128-130], and hence, a unique history of stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \dots$, all over the continuum. The histories of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \dots$ and the external body forces X, Y, Z by virtue of equations (1) imply a unique history of accelerations $\ddot{u}_x, \ddot{u}_y, \ddot{u}_z$, thereby implying a unique history of displacements u_x, u_y, u_z, all over the continuum.

If we also applied to the continuum a parallel history of arbitrary additional surface tractions $\Delta T_x, \Delta T_y, \Delta T_z$ in combination with a parallel history of such additional external body forces $\Delta X, \Delta Y, \Delta Z$ that by means of equations (1) counterbalance any developing additional stress derivatives $\Delta(\partial\sigma_{xx}/\partial x),\ \Delta(\partial\tau_{yx}/\partial y),\ \Delta(\partial\tau_{zx}/\partial z), \dots$, viz.

$$\left.\begin{aligned}
&\Delta(\partial\sigma_{xx}/\partial x) + \Delta(\partial\tau_{yx}/\partial y) + \Delta(\partial\tau_{zx}/\partial z) + \Delta X = 0\\
&\Delta(\partial\tau_{xy}/\partial x) + \Delta(\partial\sigma_{yy}/\partial y) + \Delta(\partial\tau_{zy}/\partial z) + \Delta Y = 0\\
&\Delta(\partial\tau_{xz}/\partial x) + \Delta(\partial\tau_{yz}/\partial y) + \Delta(\partial\sigma_{zz}/\partial z) + \Delta Z = 0
\end{aligned}\right\} \tag{5}$$

then, we could retain the same history of accelerations, thereby retaining the same history of displacements, as without the parallel history of additional surface tractions and additional external body forces. This possibility proves that the same history of displacements in a continuum can be related to different histories of surface tractions, and hence, to different works of surface tractions. Thus, the work of the surface tractions of the continuum, and hence, the total work of the internal stresses all over the continuum, is not a single-valued function of only the displacement distribution in the continuum.

On this base therefore, *the internal stresses all over a continuum, whether elastic or not, must be nonconservative*.

Scholium A: The uniqueness of the solution to a given external loading

To show that a given history of surface tractions T_x, T_y, T_z and external body forces X, Y, Z implies a *unique* history of internal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ and displacements u_x, u_y, u_z, we can proceed as follows [7 pp.128-130]

Let us assume that the action of a given history of surface tractions T_x, T_y, T_z and external body forces X, Y, Z may correspond with two different histories of internal stresses, i.e. $\sigma^I_{xx}, \sigma^I_{yy}, \sigma^I_{zz}, \tau^I_{xy}, \tau^I_{yz}, \tau^I_{xz}$ and $\sigma^{II}_{xx}, \sigma^{II}_{yy}, \sigma^{II}_{zz}, \tau^{II}_{xy}, \tau^{II}_{yz}, \tau^{II}_{xz}$. Either of these stress histories must satisfy the differential equations of motion (1) and the boundary stress conditions

$$\left.\begin{aligned} T_x &= \sigma_{xx} \cdot cos(nx) + \tau_{yx} \cdot cos(ny) + \tau_{zx} \cdot cos(nz) \\ T_y &= \tau_{xy} \cdot cos(nx) + \sigma_{yy} \cdot cos(ny) + \tau_{zy} \cdot cos(nz) \\ T_z &= \tau_{xz} \cdot cos(nx) + \tau_{yz} \cdot cos(ny) + \sigma_{zz} \cdot cos(nz) \end{aligned}\right\} \qquad \text{(A1)}$$

where n denotes the line of the unit normal vector at a point of the boundary surface.

By subtracting each of equations (1) formed by $\sigma_{xx}^{II}, \sigma_{yy}^{II}, \sigma_{zz}^{II}, \tau_{xy}^{II}, \tau_{yz}^{II}, \tau_{xz}^{II}$ from the corresponding one formed by $\sigma_{xx}^{I}, \sigma_{yy}^{I}, \sigma_{zz}^{I}, \tau_{xy}^{I}, \tau_{yz}^{I}, \tau_{xz}^{I}$, we obtain the following system of equations

$$\left.\begin{aligned}
&\frac{\partial\left(\sigma_{xx}^{I}-\sigma_{xx}^{II}\right)}{\partial x}+\frac{\partial\left(\tau_{yx}^{I}-\tau_{yx}^{II}\right)}{\partial y}+\frac{\partial\left(\tau_{zx}^{I}-\tau_{zx}^{II}\right)}{\partial z}=\rho\cdot\left(\frac{\partial^2 u_x^{I}}{\partial t^2}-\frac{\partial^2 u_x^{II}}{\partial t^2}\right)\\
&\frac{\partial\left(\tau_{xy}^{I}-\tau_{xy}^{II}\right)}{\partial x}+\frac{\partial\left(\sigma_{yy}^{I}-\sigma_{yy}^{II}\right)}{\partial y}+\frac{\partial\left(\tau_{zy}^{I}-\tau_{zy}^{II}\right)}{\partial z}=\rho\cdot\left(\frac{\partial^2 u_y^{I}}{\partial t^2}-\frac{\partial^2 u_y^{II}}{\partial t^2}\right)\\
&\frac{\partial\left(\tau_{xz}^{I}-\tau_{xz}^{II}\right)}{\partial x}+\frac{\partial\left(\tau_{yz}^{I}-\tau_{yz}^{II}\right)}{\partial y}+\frac{\partial\left(\sigma_{zz}^{I}-\sigma_{zz}^{II}\right)}{\partial z}=\rho\cdot\left(\frac{\partial^2 u_z^{I}}{\partial t^2}-\frac{\partial^2 u_z^{II}}{\partial t^2}\right)
\end{aligned}\right\} \qquad (A2)$$

Similarly, from equations (A1) it follows

$$\left.\begin{aligned}
&0=\left(\sigma_{xx}^{I}-\sigma_{xx}^{II}\right)\cdot cos\left(nx\right)+\left(\tau_{yx}^{I}-\tau_{yx}^{II}\right)\cdot cos\left(ny\right)+\left(\tau_{zx}^{I}-\tau_{zx}^{II}\right)\cdot cos\left(nz\right)\\
&0=\left(\tau_{xy}^{I}-\tau_{xy}^{II}\right)\cdot cos\left(nx\right)+\left(\sigma_{yy}^{I}-\sigma_{yy}^{II}\right)\cdot cos\left(ny\right)+\left(\tau_{zy}^{I}-\tau_{zy}^{II}\right)\cdot cos\left(nz\right)\\
&0=\left(\tau_{xz}^{I}-\tau_{xz}^{II}\right)\cdot cos\left(nx\right)+\left(\tau_{yz}^{I}-\tau_{yz}^{II}\right)\cdot cos\left(ny\right)+\left(\sigma_{zz}^{I}-\sigma_{zz}^{II}\right)\cdot cos\left(nz\right)
\end{aligned}\right\} \qquad (A3)$$

Using the principle of superposition, which holds true owing to the linearity of the differential equations of motion (1) and the boundary stress conditions (A1), we can take the history of the stress differences in equations (A2) and (A3) as a new history of internal stresses corresponding with zero surface tractions and zero external body forces.

Actually, there can only exist a history of zero internal stresses in the absence of surface tractions and external body forces on the base of the hypothesis of the natural state of the continuum [7 p. 130]. Therefore, the history $\sigma_{xx}^{I}, \sigma_{yy}^{I}, \sigma_{zz}^{I}, \tau_{xy}^{I}, \tau_{yz}^{I}, \tau_{xz}^{I}$ and the history $\sigma_{xx}^{II}, \sigma_{yy}^{II}, \sigma_{zz}^{II}, \tau_{xy}^{II}, \tau_{yz}^{II}, \tau_{xz}^{II}$ must coincide with a unique history $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$. This, by equations (A2), directly implies that the histories of accelerations $\ddot{u}_x^{I}, \ddot{u}_y^{I}, \ddot{u}_z^{I}$ and $\ddot{u}_x^{II}, \ddot{u}_y^{II}, \ddot{u}_z^{II}$ must coincide with a unique history of accelerations $\ddot{u}_x, \ddot{u}_y, \ddot{u}_z$, which, in turns, implies a unique history of displacements u_x, u_y, u_z.

Scholium B: Displacements uniquely defined by accelerations

To support the view underlying the above analysis that the history of accelerations uniquely defines the history of displacements in a continuum, for given initial conditions, it is worth noting that

B.1. The displacement $u(x,y,z;t)$ is the integral of the velocity $\dot{u}(x,y,z;t)$, viz.

$$u(x,y,z;t) = \int_0^t \dot{u}(x,y,z;\tau) \cdot \mathrm{d}\tau + u(x,y,z;t=0), \tag{B1}$$

since the velocity $\dot{u}(x,y,z;t)$ is the derivative of the displacement $u(x,y,z;t)$, and integration and differentiation are inverse mathematical processes.

The existence of the acceleration $\ddot{u}(x,y,z;t)$ as the derivative of the velocity $\dot{u}(t)$

assures the continuity of the velocity $\dot{u}(x,y,z;t)$, which makes the velocity $\dot{u}(x,y,z;t)$ be integrable [16 p.97].

B.2. Similarly, the velocity $\dot{u}(x,y,z;t)$ is the integral of the acceleration $\dot{u}(x,y,z;t)$,

$$\dot{u}(x,y,z;t) = \int_0^t \ddot{u}(x,y,z;\tau)\cdot \mathrm{d}\tau + \dot{u}(x,y,z;t=0)\,, \tag{B2}$$

since the acceleration $\ddot{u}(x,y,z;t)$ is the derivative of the velocity $\dot{u}(x,y,z;t)$. Assuming the continuity of the acceleration $\ddot{u}(x,y,z;t)$, or more generally that the acceleration $\ddot{u}(x,y,z;t)$ is bounded and presents only a *finite* number of discontinuities within the time domain $(0,t)$, assures its integrability [16 p.98].

B.3. Combining equations (B1) and (B2) yields

$$u(x,y,z;t) = \int_0^t \left(\int_0^\tau \ddot{u}(x,y,z;\hat{\tau})\cdot \mathrm{d}\hat{\tau} \right) \cdot \mathrm{d}\tau + t\cdot \dot{u}(x,y,z;t=0) + u(x,y,z;t=0)\,, \tag{B3}$$

where τ, $\hat{\tau}$ stand for integrand variables of the time t, and

$u(x,y,z;t=0), \dot{u}(x,y,z;t=0)$ stand for the initial conditions.

Equation (B3) means that *for given initial conditions* $u(x,y,z;t=0)$ *and* $\dot{u}(x,y,z;t=0)$, *the deformation displacement* $u(x,y,z;t)$ *of a point of a deformable body on any coor-*

dinate axis is a single-valued function of the history of the corresponding acceleration $\ddot{u}(x,y,z;t)$ *of the point from the initiation of the deformation up to time* t. In other words, *two cases of acceleration* $\ddot{u}(x,y,z;t)$ *of the point with the same history from the initiation of the deformation up to time* t *imply that the corresponding cases of displacement* $u(x,y,z;t)$ *have the same history*. This latter conclusion is what actually justifies the postulate after equations (5) that "then, we could retain the same history of accelerations, thereby retaining the same history of displacements, as without the parallel history of additional surface tractions and additional external body forces".

6. Deformation motion as source of damping

Let us now consider a continuum exclusively subjected to surface tractions, which implies that the external body forces X,Y,Z in equations (1) are zero. Then, as exposed in par. 2, the differential equations of motion (1) necessitate that the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ along the displacements u_x,u_y,u_z equal the work of the accelerating forces $\rho\cdot\ddot{u}_x,\ \rho\cdot\ddot{u}_y,\ \rho\cdot\ddot{u}_z$ along the same displacements, which in turn equals the kinetic energy of the continuum. As a consequence, for a given final configuration of the continuum, the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ done along the transition path from the initial at-rest natural configuration to the final configuration of the continuum must equal the kinetic energy of the final configuration. It is quite reasonable that two different transition paths can lead to the given final configuration but under different final velocities.

Indeed, by keeping the same initial and final surface tractions for the two transition paths and applying different intermediate surface tractions, which results in different intermediate stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$, and hence, different intermediate accelerations $\ddot{u}_x, \ddot{u}_y, \ddot{u}_z$, it is possible to have the same initial and final configurations of the continuum and different final velocities for the two transition paths.

On this base therefore, for given final displacements, the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ along the displacements u_x, u_y, u_z all over a continuum can take on different values depending on the final velocities applied. This proves that the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ of a continuum must act as nonconservative internal body forces. And since the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ by definition coincides with the total work of the unbalanced stress components $(\partial\sigma_{xx}/\partial x)\mathrm{d}x,\ (\partial\tau_{yx}/\partial y)\mathrm{d}y,\ (\partial\tau_{zx}/\partial z)\mathrm{d}z,\ldots$, it is deduced that *the unbalanced stress components, and hence, the internal stresses, of a continuum must be classified as nonconservative stresses*. Only for a motionless deformation, the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ becomes zero, which allows the internal stresses to be conservative. This finding assures that the deformation motion of a continuum, whether elastic or not, is a source of damping.

7. Static deformation as a prerequisite of conservative internal stresses

Actually, there is a unique case where the requirement of displacements defined by accelerations, as used in par. 6 for concluding the nonconservative nature of the internal stresses, cannot be fulfilled, which allows of conservative internal stresses. This case consists in restricting the differential equations of motion (1) to

$$\left.\begin{aligned}&\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0\\&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0\\&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+Z=0\end{aligned}\right\}\tag{6}$$

which describe the *static* (i.e. motionless) deformation of a nonmassles (i.e. with $\rho \neq 0$) continuum. In this case, the accelerations by definition become zero, and hence, the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ are counter-balanced by the external body forces X, Y, Z at every point of the continuum. This balance implies that the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ must equal the total work of the external body forces X, Y, Z with opposite sign. Consequently, the classification of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ as conservative or nonconservative must be the same as the classification of the external body forces X, Y, Z, which means that *a necessary condition for conservative internal stresses is static deformation*.

For a continuum exclusively subjected to surface tractions, that is, for zero external body forces $X = Y = Z = 0$, equations (2.6) are reduced to

$$\left.\begin{aligned}&\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=0\\&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}=0\\&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0\end{aligned}\right\}\tag{7}$$

Equations (7) imply that the work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$ all over the continuum is zero, which, by virtue of equation (2), results in

$$\textit{Total work of conservative internal stresses} = \textit{strain energy}, \quad \textit{for}\ X = Y = Z = 0. \qquad (8)$$

It is noticed that equations (7) and (8) are always valid for the massless springs that connect the rigid lumped masses of a discrete system, since these springs constitute continua with zero mass density $\rho = 0$ and zero external body forces $X = Y = Z = 0$.

8. Critical points on the classical view of elastic stresses

For an elastic continuum obeying the generalized Hooke's law, the internal stresses at a point are linear functions of only the strains at the point [3 pp.97-100 eq.(11)], [7 pp.78-79 eqs.(3.23),(3.27)], which can be expressed in the matrix formulation [15 p.16]

$$\boldsymbol{\sigma}(x,y,z;t) = \boldsymbol{\kappa}(x,y,z)\cdot\boldsymbol{\varepsilon}(x,y,z;t), \qquad (9)$$

where $\boldsymbol{\sigma}(x,y,z;t)$, $\boldsymbol{\varepsilon}(x,y,z;t)$ and $\boldsymbol{\kappa}(x,y,z)$ stand for the stress tensor as the column matrix of the balanced stress components $\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{yz},\tau_{xz}$, the strain tensor as the column matrix of the strains $\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\gamma_{xy},\gamma_{yz},\gamma_{xz}$, and the square matrix of constant elastic coefficients, respectively, at the point (x,y,z) of the elastic continuum.

The strains $\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\gamma_{xy},\gamma_{yz},\gamma_{xz}$ at a point are defined as the space derivatives of the displacement distribution at the point, thereby being single-valued functions of the displacement distribution in the continuum. By the generalized Hooke's law (9), the same must hold true for the balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$. And besides, owing to the linearity of the law, the work done by all balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ of a point along its strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$ proves to be a single valued function of the strains [5 pp.244-246 eq.(132)], and hence, of the displacement distribution. Consequently, the total work of the balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ for all points of an elastic continuum, i.e. the strain energy of the continuum, must be a single-valued function of the strains [5 p.247 eq.(135)], and hence, of the displacement distribution in the continuum. Accordingly, in view of the notion of conservative internal stresses discussed in par. 3, the elastic balanced stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ must be conservative. This latter, on account of the nonconservative nature of the internal stresses shown in par. 5, leads to the conclusion obtained in par. 6 that *the stress derivatives* $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$ *of an elastic continuum obeying the generalized Hooke's law (9) must be nonconservative*.

Indeed, each of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$ at a point does work along any displacement component of the point that cannot be expressed in terms of only the stress derivative or the displacement component, because these two latter magnitudes at a point are not single-valued functions of each other. Thus, the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$ all over an elastic continuum cannot be a single-valued function of the displacement distribution in the continuum, which verifies that the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$ are nonconservative.

It is worth noting that the necessary condition (8) for conservative internal stresses in a continuum exclusively subjected to surface tractions and the classical definition of conservative stresses [cf par. 3] require that the strain energy become a single-valued function of only the displacement distribution in the continuum. This requirement, as shown above, is satisfied by the generalized Hooke's law (9), which justifies why the law leads to conservative internal stresses on the assumption of zero work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$, as expressed by either of equations (7) and (8). Only on this restricted assumption can stand the validity of the classical view that the generalized Hooke's law (9) leads to conservative internal stresses.

9. Principle of virtual work, energy losses and the first thermodynamic axiom

There is a widely-spread illusion that on account of the principle of virtual work (i.e. the work of external body forces and surface tractions equals the sum of the corresponding strain energy and kinetic energy) the deformation of a continuum is inconsistent with energy losses, and hence, with damping, which seems to contradict our analysis.

Actually, the forces and stresses in any continuum undergoing dynamic deformation, whether elastic or not, are ruled by Newton's second axiom irrespective of their conservative or nonconservative character. And the energy equivalent of Newton's second axiom is the principle of virtual work expressed as below [15 pp. 264-267 eq.(10.3)]

$$\delta W(t)=\delta U_{in}(t)+\iiint_V \rho(x,y,z)\cdot\delta\mathbf{u}^T(x,y,z;t)\cdot\ddot{\mathbf{u}}(x,y,z;t)\cdot \mathrm{d}V\,, \tag{10}$$

where $\rho(x,y,z)$ stands for the mass density at the point (x,y,z) of the system.

V stands for the total volume of the system.

T as an upper index stands for the operator of transposing a matrix.

δ stands for the operator of *virtual variations*.

$\delta W(t)$, $\delta U_{in}(t)$ and $\iiint_V \rho(x,y,z) \cdot \delta \mathbf{u}^T(x,y,z;t) \cdot \ddot{\mathbf{u}}(x,y,z;t) \cdot \mathrm{d}V$ stand for the virtual work of the external body forces and surface tractions, the virtual strain energy and the virtual kinetic energy, respectively, all over the continuum.

By definition, the virtual strain energy $\delta U_{in}(t)$ equals [15 p.267 eq.(10.4)]

$$\delta U_{in}(t) = \iiint_V \delta \boldsymbol{\varepsilon}^T(x,y,z;t) \cdot \boldsymbol{\sigma}(x,y,z;t) \cdot \mathrm{d}V \,, \tag{11}$$

and the virtual work $\delta W(t)$ equals [15 p.267 eq.(10.5)]

$$\delta W(t) = \iiint_V \delta \mathbf{u}^T(x,y,z;t) \cdot \mathbf{X}(x,y,z;t) \cdot \mathrm{d}V + \iint_S \delta \mathbf{u}^T(x,y,z;t) \cdot \mathbf{T}(x,y,z;t) \cdot \mathrm{d}S \,, \tag{12}$$

with $\mathbf{X}(x,y,z;t)$ and $\mathbf{T}(x,y,z;t)$ denoting the column matrix of external body forces X,Y,Z at a point of the continuum and the column matrix of surface tractions (i.e. external stresses) at a point of the boundary surface S of the continuum, respectively, and

$\mathbf{u}(x,y,z;t)$, $\boldsymbol{\varepsilon}(x,y,z;t)$, $\boldsymbol{\sigma}(x,y,z;t)$ exclusively resulting from these loads.

The principle of virtual work (10) applies to any continuum ruled by Newton's second axiom and small deformations, and exclusively refers to the action of external body forces and surface tractions [3 pp.93-95 eq.(6)]. It can serve as the complete energy balance underlying the dynamic behaviour of any continuum subjected to external body forces and surface tractions, on the assumption that no energy form other than that of the work of external body forces and surface tractions can enter or escape from the continuum. Thus, the principle of virtual work (10) allows the study of the dynamic behaviour of the continuum without any recourse to heat losses of the continuum.

Surprisingly, the energy balance expressed by the principle of virtual work (10) is conventionally deemed to be inconsistent with energy losses, and hence, with damping. This is due to the incorrect view of energy losses as a difference between the left-hand and the right-hand members of the principle of virtual work (10), despite that the equality of the two members results from Newton's second axiom, which holds true even for nonconservative forces and stresses. In fact, the energy losses of a continuum represent but the differences between the values of the work done by the internal stresses, which, recalling par. 4, equal the differences between the values of the work done by the surface tractions, along loading and unloading the continuum. And this equality of differences assures that the losses of the work of internal stresses escape from the continuum in the form of work of surface tractions. In short, the energy losses are due to the hysteresis loops caused by the multi-valuedness of the work of internal stresses, and hence, recalling par. 4, of the work of surface tractions, for given strains or displacements.

On this base therefore, the principle of virtual work (10) must be faced as a particular form of the first thermodynamic axiom whose both members can include energy losses or gains in the form of work of stresses only, thereby being consistent with damping.

The first thermodynamic axiom introduces two additional classical magnitudes:

i) The *heat* $Q(t)$ *externally supplied to the natural state of the continuum until time* t.

ii) The *internal energy* $\Theta(t)$ *of the continuum*, which is defined as the total energy of the continuum [17 p.4], that is, the sum of the supplied work of all external body forces and surface tractions plus the supplied heat. The latter is transformed into work of thermal internal stresses resulting from the temperature differences added to the natural-state temperature of the continuum by the supplied heat and into change of the internal energy content of the natural state of the continuum.

Then, the first thermodynamic axiom can be expressed by the formula [17 pp.4-5]

$$\delta\Theta(t) = \delta Q(t) + \delta W(t), \tag{13}$$

which by virtue of the principle of virtual work (10) implies the equality

$$\delta\Theta(t) = \delta Q(t) + \delta U_{in}(t) + \iiint_V \rho(x,y,z) \cdot \delta\mathbf{u}^T(x,y,z;t) \cdot \ddot{\mathbf{u}}(x,y,z;t) \cdot \mathrm{d}V \,. \tag{14}$$

Two interesting corollaries can be deduced from equations (10) and (14):

1. When $\delta Q(t) = 0$, then $\delta\Theta(t) = \delta U_{in}(t) + \iiint_V \rho(x,y,z) \cdot \delta\mathbf{u}^T(x,y,z;t) \cdot \ddot{\mathbf{u}}(x,y,z;t) \cdot \mathrm{d}V$, and the first thermodynamic axiom (13) reduces to the principle of virtual work (10).
2. When $\delta Q(t) \neq 0$, then $\delta\Theta(t) \neq \delta U_{in}(t) + \iiint_V \rho(x,y,z) \cdot \delta\mathbf{u}^T(x,y,z;t) \cdot \ddot{\mathbf{u}}(x,y,z;t) \cdot \mathrm{d}V$, which means that the internal energy $\delta\Theta(t)$ includes an equal to $\delta Q(t)$ total of strain and kinetic energy done by thermal internal stresses plus a change of the internal energy content of the natural state of the continuum, in addition to the mechanical energy $\delta U_{in}(t) + \iiint_V \rho(x,y,z) \cdot \delta\mathbf{u}^T(x,y,z;t) \cdot \ddot{\mathbf{u}}(x,y,z;t) \cdot \mathrm{d}V$. Hence, any heat exchanges be-

tween the continuum and its external environment will have equal thermal effects on the internal energy $\Theta(t)$ of the continuum, thereby resulting in thermal differences from the mechanical energy determined exclusively within the frame of the principle of virtual work (10). In studying these thermal effects and the resulting stress and strain differences of the continuum consists the role of the first thermodynamic axiom (13).

All in all, the first thermodynamic axiom (13) and the principle of virtual work (10) can both account for energy losses, with the difference that the former allows heat to be added to the work of external body forces and surface tractions and to the energy losses in the form of work of stresses that characterize the latter.

10. Conclusions

The total work of the internal stresses developed in a continuum subjected to dynamic loading, whether elastic or not, proves to be not a single-valued function of only the displacement distribution all over the continuum, which means that the internal stresses are nonconservative, thereby including damping components. This indicates damping as an inherent effect in the continuum model of dynamics, whether elastic or not.

Actually, the total work of the internal stresses of a continuum does not coincide with the strain energy of the continuum, but, instead, equals the sum of the strain energy of the continuum plus the work of the internal body forces formed by the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z,\ldots$, with this latter work exclusively contributing to the formation of the kinetic energy of the continuum. And what implies the nonconservative nature of the internal stresses of an elastic continuum undergoing a dynamical deformation is that, in spite of the strain energy, the total work of the stress derivatives $\partial\sigma_{xx}/\partial x,\ \partial\tau_{yx}/\partial y,\ \partial\tau_{zx}/\partial z, \ldots$ cannot be a single-valued function of only the displacement distribution all over the continuum.

Conservative internal stresses can only develop for static (i.e. motionless) deformation of a nonmassles continuum. The static deformation for the case of a continuum exclusively subjected to surface tractions implies zero work of stress derivatives.

References

[1] T. W. B. Kimble & F. H. Berkshire, *Classical Mechanics*, 4th edition, Longman, 1996.

[2] J. R. Willis, C. Bailey, J. Jones, *Examiners' Reports to Heriot-Watt University*, 2006, [my personal correspondence].

[3] A. E. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th edition, Dover, 1944.

[4] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd edition, McGraw-Hill, 1956.

[5] S. P. Timoshenko & J. N. Goodier, *Theory of Elasticity*, 3rd edition, McGraw-Hill, 1970.

[6] A. E. Armenàkas, *Classical Structural Analysis*, McGraw-Hill, 1988.

[7] M. Filonenko-Borodich, *Theory of Elasticity*, Dover, 1965.

[8] J. E. Gordon, *Structures or Why Things Don't Fall Down*, Penguin Books, 1978.

[9] E. P. Popov, *Engineering Mechanics of Solids*, Prentice Hall, 1999.

[10] H. Goldstein, *Classical Mechanics*, Addison-Wesley, 1980.

[11] I. S. Sokolnikoff & R. M. Redheffer, *Mathematics of Physics and Modern Engineering*, 2nd edition, McGraw-Hill, 1966.

[12] W. Kaplan, *Advanced Calculus*, 2nd edition, Addison-Wesley, 1973.

[13] T. von Karman & M. A. Biot, *Mathematical Methods in Engineering*, McGraw-Hill, 1940.

[14] L. Meirovitch, *Elements of Vibration Analysis*, 2nd edition, McGraw-Hill, 1986.

[15] J. S. Przemeniecki, *Theory of Matrix Structural Analysis*, McGraw-Hill, 1968.

[16] H. S. Carslaw, *An Introduction to the Theory of Fourier's Series and Integrals*, 3rd edition, Dover, 1950.

[17] D. Chandler, *Introduction to Modern Statistical Mechanics*, Oxford University Press, 1987.