Technical Note

Inverse kinematic analysis for triple-octahedron variable-geometry truss manipulators

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Abstract: In this paper, a new triple-octahedron variable-geometry truss manipulator is presented. Its inverse kinematic solutions in closed form are studied. An input–output displacement equation in one output variable is derived. The solution procedure is given in detail. A numerical example is illustrated.

Keywords: inverse kinematics, triple-octahedron variable-geometry truss manipulator, closed-form solution, robot manipulator

NOTATION

a_i, b_i, c_i, d_i, e_i	coefficients of fourth-order polynomial equations ($i = 1, 2$)
$\mathrm{A_i}$, $\mathrm{B_i}$, $\mathrm{C_i}$	joint points of the triple-octahedron, variable-geometry truss manipulator ($i = 1, 2, 3, 4$)
$\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}_i$	position vectors of points in the fixed coordinate system
c, s	cosine and sine mathematical functions
G, H, D, E	projective points of perpendicular in the triangular planes
k_{ij}	coefficients of the functions between known geometrical parameters and input parameters ($i, j = 1, 2, \ldots, 6$)
l_i	lengths of six extensible links (actuator members) ($i = 1, 2, \ldots, 6$)
m_i	lengths of inextensible links ($i = 1, 2, \ldots, 6$)
M, M_i	normals of the triangular planes ($i = 1, 2, 3$)
N, N_i	parallel line of the coordinate axes ($i = 1, 2, 3$)
$oxyz$	moving coordinate system
O, O'_i, O''_i	foot of perpendicular in the triangular planes ($i = 1, 2, 3, 4$)
$OXYZ$	fixed coordinate system
p_i	coefficients of 16th-order polynomial equations ($i = 1, 2, \ldots, 16$)
q_i	coefficients of fourth-order polynomial equations for y_3 ($i = 1, 2, 3, 4, 5$)
[**T**]	input–output displacement transform (the homogeneous transformation matrix from coordinate system $o_1x_1y_1z_1$ to coordinate system $OXYZ$)
x_i, y_i	coordinate values of joint points
θ_i	dihedral angles between the end-effect platform place and moving actuated planes ($i = 1, 2, 3$)
φ_i	joint angles of links within the triangular planes ($i = 1, 2, 3$)
ψ_i	dihedral angles between the base platform place $A_1B_1C_1$ and the middle actuated planes ($i = 1, 2, 3$)

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1 INTRODUCTION

A variable-geometry truss mechanism (VGTM) is a statically determinate truss that has been modified to contain some number of variable length links. VGTMs have very good stiffness–weight ratios and are theoretically composed of two force links. No bending moments or torques can be transmitted at the joints. Moreover, they can be designed to be collapsible. These characteristics give VGTMs potential applications, discussed by Arun *et al.* [**1**], such as beams to position equipment in space, supports for space antenna, berthing devices and manipulator arms. As a robot manipulator, VGTMs have higher stiffness than serial link manipulators and a large workspace compared with parallel ones. Therefore, they are considered as a new type of robot manipulator.

All variable-geometry truss mechanisms are made up of some combination of fundamental units, such as the tetrahedron, octahedron, decahedron and odecahedron. The solution to the position analysis problem of VGTMs can be carried out using a number of different approaches. In recent years, a number of fruitful investigations have been made to explore position analysis problems for VGTMs [**1**–**8**]. The authors applied a homotopy continuation algorithm to solve the inverse displacement analysis problem of triple-octahedron, variable-geometry truss manipulators [**9**]. Although all the possible solutions can be found, the computation is expensive. A closed-form inverse displacement analysis by an elimination method will provide more information about the geometry and kinematic behaviour of manipulators, and this information is also extremely useful in practice for the control of manipulators. In this paper, inverse displacement analysis in closed form is implemented for triple-octahedron, variable-geometry truss manipulators by using the elimination method. A 128th-degree algebraic equation in one output variable is derived.

2 CONSTRAINT EQUATIONS

A six-degree-of-freedom (6 DOF), triple-octahedron, variable-geometry truss manipulator is represented schematically in Fig. 1. The manipulator consists of three octahedra $A_{i+1}B_{i+1}C_{i+1} - A_iB_iC_i \quad (i = 1, 2, 3)$ stacked upon one another. This includes an end-effector platform $A_4B_4C_4$, a base platform $A_1B_1C_1$ and two middle actuated planes in which six extensible links are located respectively. Referring to Fig. 1, the fixed coordinate system $oxyz$ is rigidly attached to the base platform so that the z axis coincides with the normal to the base face and the x axis aligns with line A_1B_1. The moving coordinate system $o_1x_1y_1z_1$ is attached to the top triangular face so that the z_1 axis coincides with the normal to the top face and the x_1 axis is aligned with line B_4C_4. Let ψ_1, ψ_2, ψ_3 denote respectively the dihedral angles between planes $A_1B_1B_2$, $A_1C_1A_2$, $B_1C_1C_2$ and plane $A_1B_1C_1$, and $\theta_1, \theta_2, \theta_3$ denote respectively the dihedral angles between planes $B_4C_4B_3$, $A_4B_4A_3$, $A_4C_4C_3$ and plane $A_4B_4C_4$. The lines B_2O, A_2O' and C_2O'' are perpendicular to lines A_1B_1, A_1C_1 and B_1C_1 respectively, and lines B_3O_1, $A_3O'_1$ and $C_3O''_1$ are orthogonal to lines B_4C_4, A_4B_4 and A_4C_4 respectively.

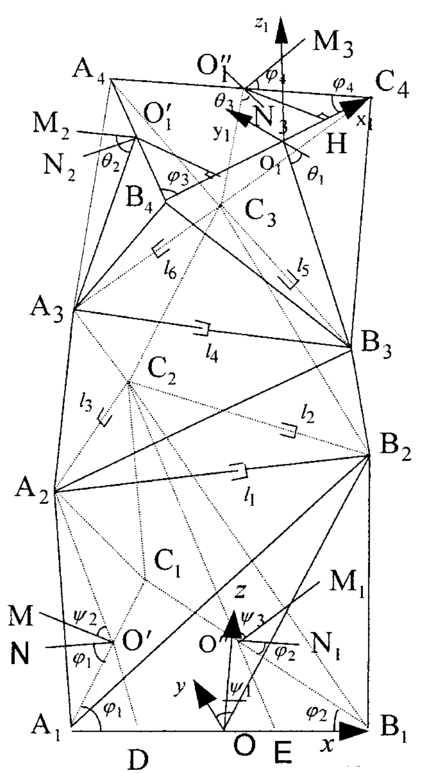

Fig. 1 A six-degree-of-freedom, triple-octahedron, variable-geometry truss manipulator

According to the coordinate system established above, the position vectors of points A_2, B_2 and C_2 in the fixed coordinate system $oxyz$ can be written as follows:

$$A_2 = \begin{bmatrix} -O'A_2 c\psi_2 s\varphi_1 - O'D \\ O'A_2 c\psi_2 c\varphi_1 + O'D \\ O'A_2 s\psi_2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ OB_2 c\psi_1 \\ OB_2 s\psi_1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} O''C_2 c\psi_3 s\varphi_2 + O''E \\ O''C_2 c\psi_3 c\varphi_2 + O''E \\ O''C_2 s\psi_3 \end{bmatrix} \tag{1}$$

The position vectors of points A_3, B_3 and C_3 in the moving coordinate system $o_1x_1y_1z_1$ can be derived and expressed in the fixed coordinate system $oxyz$ as follows:

$$\begin{bmatrix} A_3 \\ 1 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} -A_3O'_1 c\theta_2 s\varphi_3 - O'_1G \\ A_3O'_1 c\theta_2 c\varphi_3 + O'_1G \\ A_3O'_1 s\theta_2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} B_3 \\ 1 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} 0 \\ O_1B_3 c\theta_1 \\ O_1B_3 s\theta_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{C}_3 \\ 1 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} \mathrm{O}_1''\mathrm{C}_3\mathrm{c}\theta_3\mathrm{s}\varphi_4 + \mathrm{O}_1''\mathrm{H} \\ \mathrm{O}_1''\mathrm{C}_3\mathrm{c}\theta_3\mathrm{c}\varphi_4 + \mathrm{O}_1''\mathrm{H} \\ \mathrm{O}_1''\mathrm{C}_3\mathrm{s}\theta_3 \\ 1 \end{bmatrix} \tag{2}$$

The constraint equations of the reverse displacement analysis problem of triple-octahedron VGTMs can be written as follows:

$$\begin{aligned} &(\boldsymbol{B}_3 - \boldsymbol{B}_2)^{\mathrm{T}}(\boldsymbol{B}_3 - \boldsymbol{B}_2) = m_1^2 \\ &(\boldsymbol{C}_3 - \boldsymbol{B}_2)^{\mathrm{T}}(\boldsymbol{C}_3 - \boldsymbol{B}_2) = m_2^2 \\ &(\boldsymbol{C}_3 - \boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{C}_3 - \boldsymbol{C}_2) = m_3^2 \\ &(\boldsymbol{A}_3 - \boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{A}_3 - \boldsymbol{C}_2) = m_4^2 \\ &(\boldsymbol{A}_3 - \boldsymbol{A}_2)^{\mathrm{T}}(\boldsymbol{A}_3 - \boldsymbol{A}_2) = m_5^2 \\ &(\boldsymbol{B}_3 - \boldsymbol{A}_2)^{\mathrm{T}}(\boldsymbol{B}_3 - \boldsymbol{A}_2) = m_6^2 \end{aligned} \tag{3}$$

where $m_1, m_2, \ldots, m_6$ are the lengths of the fixed-length links $\mathrm{B_2B_3}$, $\mathrm{B_2C_3}$, $\mathrm{C_2C_3}$, $\mathrm{C_2A_3}$, $\mathrm{A_2A_3}$ and $\mathrm{A_2B_3}$ respectively. After substitution of equations (1) and (2) and triangular identity $\mathrm{c}\psi_i = (1 - x_i^2)/(1 + x_i^2)$, $\mathrm{s}\psi_i = (2x_i)/(1 + x_i^2)$, $\mathrm{c}\theta_i = (1 - y_i^2)/(1 + y_i^2)$, $\mathrm{s}\theta_i = (2y_i)/(1 + y_i^2)$ $(i = 1, 2, 3)$ into equations (3) and rearrangement, the following are obtained:

$$\begin{aligned} &\left(k_{11}y_1^2 + k_{12}y_1 + k_{13}\right)x_1^2 + \left(k_{14}y_1^2 + k_{15}y_1 + k_{16}\right)x_1 \\ &\quad + \left(k_{17}y_1^2 + k_{18}y_1 + k_{19}\right) = 0 \\ &\left(k_{21}y_3^2 + k_{22}y_3 + k_{23}\right)x_1^2 + \left(k_{24}y_3^2 + k_{25}y_3 + k_{26}\right)x_1 \\ &\quad + \left(k_{27}y_3^2 + k_{28}y_3 + k_{29}\right) = 0 \\ &\left(k_{31}y_3^2 + k_{32}y_3 + k_{33}\right)x_3^2 + \left(k_{34}y_3^2 + k_{35}y_3 + k_{36}\right)x_3 \\ &\quad + \left(k_{37}y_3^2 + k_{38}y_3 + k_{39}\right) = 0 \\ &\left(k_{41}y_2^2 + k_{42}y_2 + k_{43}\right)x_3^2 + \left(k_{44}y_2^2 + k_{45}y_2 + k_{46}\right)x_3 \\ &\quad + \left(k_{47}y_2^2 + k_{48}y_2 + k_{49}\right) = 0 \\ &\left(k_{51}y_2^2 + k_{52}y_2 + k_{53}\right)x_2^2 + \left(k_{54}y_2^2 + k_{55}y_2 + k_{56}\right)x_2 \\ &\quad + \left(k_{57}y_2^2 + k_{58}y_2 + k_{59}\right) = 0 \\ &(\mathrm{k}_{61}y_1^2 + k_{62}y_1 + k_{63})x_2^2 + \left(k_{64}y_1^2 + k_{65}y_1 + k_{66}\right)x_2 \\ &\quad + \left(k_{67}y_1^2 + k_{68}y_1 + k_{69}\right) = 0 \end{aligned} \tag{4}$$

3 ELIMINATION OF EQUATION

Equation (4) can be rewritten in the following form:

$$u_1x_1^2 + v_1x_1 + w_1 = 0 \tag{5}$$

$$u_2x_1^2 + v_2x_1 + w_2 = 0 \tag{6}$$

$$u_3x_3^2 + v_3x_3 + w_3 = 0 \tag{7}$$

$$u_4x_3^2 + v_4x_3 + w_4 = 0 \tag{8}$$

$$u_5x_2^2 + v_5x_2 + w_5 = 0 \tag{9}$$

$$u_6x_2^2 + v_6x_2 + w_6 = 0 \tag{10}$$

Multiplying equations (5) and (6) by x_1, two additional equations are obtained. The total four equations can be represented by the following matrix form:

$$\begin{bmatrix} u_1 & v_1 & w_1 & 0 \\ 0 & u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 & 0 \\ 0 & u_2 & v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 \\ x_1 \\ 1 \end{bmatrix} = 0 \tag{11}$$

The necessary and sufficient condition of existence of a non-zero solution for equation (11) is that the determinant of the coefficient matrix is equal to zero. This results in the following polynomial equation:

$$q_1\,y_1^4 + q_2\,y_1^3 + q_3\,y_1^2 + q_4\,y_1 + q_5 = 0 \tag{12}$$

where coefficients q_i $(i = 1, 2, \ldots, 5)$ are not higher than fourth-order polynomials about y_3. Similarly, eliminating x_3^2 and x_3 from equations (7) and (8) and x_2^2 and x_2 from equations (9) and (10) respectively gives

$$a_1\,y_2^4 + b_1\,y_2^3 + c_1\,y_2^2 + d_1\,y_2 + e_1 = 0 \tag{13}$$

$$a_2\,y_2^4 + b_2\,y_2^3 + c_2\,y_2^2 + d_2\,y_2 + e_2 = 0 \tag{14}$$

where a_1, b_1, c_1, d_1 and e_1 are all the polynomials about y_3, the order of which is not higher than 4, and a_2, b_2, c_2, d_2 and e_2 are all the polynomials about y_1, the order of which is not higher than 4.

Equations (13) and (14) can be grouped into four sets. Eliminating y_2^4, y_2^3, y_2^2 and y_2 from each of the equations gives the following four cubic equations [10]:

$$\begin{aligned}
&(ab)y_2^3 + (ac)y_2^2 + (ad)y_2 + (ae) = 0\\
&(ac)y_2^3 + [(ad) + (bc)]\, y_2^2 + [(ae) + (bd)]\, y_2 + (be) = 0\\
&(ad)y_2^3 + [(ae) + (bd)]\, y_2^2 + [(be) + (cd)]\, y_2 + (ce) = 0\\
&(ae)y_2^3 + (be)y_2^2 + (ce)y_2 + (de) = 0
\end{aligned} \tag{15}$$

from which it is possible to form the following system of equations:

$$\begin{bmatrix} (ab) & (ac) & (ad) & (ae) \\ (ac) & (ad)+(bc) & (ae)+(bd) & (be) \\ (ad) & (ae)+(bd) & (be)+(cd) & (ce) \\ (ae) & (be) & (ce) & (de) \end{bmatrix} \begin{bmatrix} y_2^3 \\ y_2^2 \\ y_2 \\ 1 \end{bmatrix} = 0 \tag{16}$$

where $(ab) = a_1b_2 - a_2b_1$, etc.

By making the determinant of the coefficient matrix equal to zero, the following equation is obtained:

$$p_1 y_1^{16} + p_2 y_1^{15} + p_3 y_1^{14} + \cdots + p_{16} y_1 + p_{17} = 0 \tag{17}$$

where p_i $(i = 1, 2, \ldots, 17)$ are not higher than 16th-order polynomials about y_3.

Multiplying equation (17) separately by y_1, y_1^2, y_1^3 and equation (12) separately by $y_1, y_1^2, \ldots, y_1^{14}, y_1^{15}$ gives 20 equations in matrix form as follows:

$$\begin{bmatrix}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & \cdots & p_{16} & p_{17} & 0 & 0 & 0 \\
0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & \cdots & p_{15} & p_{16} & p_{17} & 0 & 0 \\
0 & 0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & \cdots & p_{14} & p_{15} & p_{16} & p_{17} & 0 \\
0 & 0 & 0 & p_1 & p_2 & p_3 & p_4 & p_5 & \cdots & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} \\
q_1 & q_2 & q_3 & q_4 & q_5 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & q_1 & q_2 & q_3 & q_4 & q_5 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_1 & q_2 & q_3 & q_4 & q_5 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_1 & q_2 & q_3 & q_4 & q_5 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_1 & q_2 & q_3 & q_4 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & \cdots & & & & & \\
 & & & & & & & & \cdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & q_1 & q_2 & q_3 & q_4 & q_5
\end{bmatrix}
\begin{bmatrix} y_1^{19} \\ y_1^{18} \\ y_1^{17} \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ y_1^2 \\ y_1 \\ 1 \end{bmatrix} = 0 \tag{18}$$

For equation (18) to have a non-trivial solution, the determinant of the coefficient matrix is set equal to zero, and thus an output displacement equation containing only one variable y_3 is obtained. This is a 128th-order algebraic equation about y_3. For each value of y_3, the corresponding y_1 can be obtained from equation (18), y_2 from equation (16) and x_1 from (11). Similarly to the computation of x_1, the variables $x_2$and x_3 can also be found.

As soon as x_i and y_i $(i = 1, 2, 3)$ are found, ψ_1, ψ_2, ψ_3 and $\theta_1, \theta_2, \theta_3$ can be evaluated from triangular formulae, and then the position vectors of points A_i, B_i, C_i $(i = 2, 3)$ in the *oxyz* coordinate system can be computed by substituting ψ_1, ψ_2, ψ_3 and $\theta_1, \theta_2, \theta_3$ into equations (1) and (2). Furthermore, the lengths of actuator members l_i $(i = 1, 2, \ldots, 6)$ can be found.

4 NUMERICAL EXAMPLE

A triple-octahedron VGTM is taken as an example to explain the method. The lengths of fixed-length links, each side of the end-effector triangular platform and each side of the base triangular platform, are all 30 mm, i.e.

$$\begin{aligned}
&A_iB_i = B_iC_i = A_iC_i = 30\,\text{mm}, && i = 1, 4\\
&A_iA_{i+1} = B_iB_{i+1} = C_iC_{i+1} = 30\,\text{mm}, && i = 1, 2, 3\\
&A_iB_{i+1} = B_iC_{i+1} = C_iA_{i+1} = 30\,\text{mm}, && i = 1, 2, 3
\end{aligned}$$

Table 1 Eighteen sets of real roots for equation (4)

	x_1	x_2	x_3	y_1	y_2	y_3
1	0.752	1.991	1.467	−2.936	−3.991	−0.650
2	0.258	1.994	1.497	−0.389	−3.988	−0.650
3	0.707	1.414	1.414	−0.271	−4.253	−0.627
4	0.189	2.615	1.681	−0.493	−3.202	−0.738
5	0.144	4.620	3.544	−0.707	−1.414	1.179
6	0.707	1.414	1.414	−2.708	−4.253	−7.335
7	0.144	0.477	0.347	−0.673	−1.402	−1.419
8	0.144	0.483	0.346	−0.707	−1.414	−1.414
9	1.627	0.483	3.544	−0.707	−1.414	−1.414
10	0.707	1.414	1.449	−0.271	−0.446	−0.738
11	0.707	1.414	1.414	−2.708	−0.446	−7.335
12	0.707	1.414	1.414	−0.271	−0.446	−0.627
13	1.806	0.500	3.429	−0.783	−1.456	−2.535
14	1.627	4.620	3.544	−0.707	−1.414	−1.414
15	0.144	4.620	0.346	−0.707	−1.414	−1.414
16	1.627	0.483	0.346	−0.707	−1.414	−1.414
17	1.578	4.367	0.323	−0.686	−1.853	−1.347
18	0.144	0.483	3.544	−0.707	−1.414	−1.414

The position and orientation of the end-effector are given below:

$$[\mathbf{T}] = \begin{bmatrix} 0.967 & -0.259 & 0 & 6 \\ 0.259 & 0.967 & 0 & 7 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All 128 sets of roots for equations (4) are obtained by running a program on the computer. The solutions are verified. Eighteen sets of real roots are listed in Table 1.

5 CONCLUSIONS

In this paper, closed-form solutions for the inverse kinematic analysis of a triple-octahedron, variable-geometry truss manipulator are presented for the first time. A 128th-degree algebraic equation in one unknown is derived. A numerical example is tested. The results show the method is simple, effective and accurate. In the experimental computation, no extraneous roots are found.

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