

# Inverse kinematic analysis for triple-octahedron variable-geometry truss manipulators

L J Xu<sup>1</sup>, G Y Tian<sup>2\*</sup>, Y Duan<sup>1</sup> and S X Yang<sup>1</sup>

<sup>1</sup>School of Manufacturing Engineering, Sichuan University, Chengdu, People's Republic of China

<sup>2</sup>School of Engineering, University of Huddersfield, UK

**Abstract:** In this paper, a new triple-octahedron variable-geometry truss manipulator is presented. Its inverse kinematic solutions in closed form are studied. An input–output displacement equation in one output variable is derived. The solution procedure is given in detail. A numerical example is illustrated.

**Keywords:** inverse kinematics, triple-octahedron variable-geometry truss manipulator, closed-form solution, robot manipulator

## NOTATION

$a_i, b_i, c_i, d_i, e_i$	coefficients of fourth-order polynomial equations ( $i = 1, 2$ )
$A_i, B_i, C_i$	joint points of the triple-octahedron, variable-geometry truss manipulator ( $i = 1, 2, 3, 4$ )
$A_i, B_i, C_i$	position vectors of points in the fixed coordinate system
$c, s$	cosine and sine mathematical functions
$G, H, D, E$	projective points of perpendicular in the triangular planes
$k_{ij}$	coefficients of the functions between known geometrical parameters and input parameters ( $i, j = 1, 2, \dots, 6$ )
$l_i$	lengths of six extensible links (actuator members) ( $i = 1, 2, \dots, 6$ )
$m_i$	lengths of inextensible links ( $i = 1, 2, \dots, 6$ )
$M, M_i$	normals of the triangular planes ( $i = 1, 2, 3$ )
$N, N_i$	parallel line of the coordinate axes ( $i = 1, 2, 3$ )
$oxyz$	moving coordinate system
$O, O'_i, O''_i$	foot of perpendicular in the triangular planes ( $i = 1, 2, 3, 4$ )
$OXYZ$	fixed coordinate system
$p_i$	coefficients of 16th-order polynomial equations ( $i = 1, 2, \dots, 16$ )

$q_i$	coefficients of fourth-order polynomial equations for $y_3$ ( $i = 1, 2, 3, 4, 5$ )
$[T]$	input–output displacement transform (the homogeneous transformation matrix from coordinate system $o_1x_1y_1z_1$ to coordinate system $OXYZ$ )
$x_i, y_i$	coordinate values of joint points
$\theta_i$	dihedral angles between the end-effect platform plane and moving actuated planes ( $i = 1, 2, 3$ )
$\varphi_i$	joint angles of links within the triangular planes ( $i = 1, 2, 3$ )
$\psi_i$	dihedral angles between the base platform plane $A_1B_1C_1$ and the middle actuated planes ( $i = 1, 2, 3$ )

## 1 INTRODUCTION

A variable-geometry truss mechanism (VGTM) is a statically determinate truss that has been modified to contain some number of variable length links. VGTMs have very good stiffness–weight ratios and are theoretically composed of two force links. No bending moments or torques can be transmitted at the joints. Moreover, they can be designed to be collapsible. These characteristics give VGTMs potential applications, discussed by Arun *et al.* [1], such as beams to position equipment in space, supports for space antenna, berthing devices and manipulator arms. As a robot manipulator, VGTMs have higher stiffness than serial link manipulators and a large workspace compared with parallel ones. Therefore, they are considered as a new type of robot manipulator.

The MS was received on 15 September 1999 and was accepted after revision for publication on 20 April 2000.

\*Corresponding author: School of Engineering, University of Huddersfield, Queensgate, Huddersfield HD1 3DH, UK.

All variable-geometry truss mechanisms are made up of some combination of fundamental units, such as the tetrahedron, octahedron, decahedron and odecadron. The solution to the position analysis problem of VGTM's can be carried out using a number of different approaches. In recent years, a number of fruitful investigations have been made to explore position analysis problems for VGTM's [1-8]. The authors applied a homotopy continuation algorithm to solve the inverse displacement analysis problem of triple-octahedron, variable-geometry truss manipulators [9]. Although all the possible solutions can be found, the computation is expensive. A closed-form inverse displacement analysis by an elimination method will provide more information about the geometry and kinematic behaviour of manipulators, and this information is also extremely useful in practice for the control of manipulators. In this paper, inverse displacement analysis in closed form is implemented for triple-octahedron, variable-geometry truss manipulators by using the elimination method. A 128th-degree algebraic equation in one output variable is derived.

2 CONSTRAINT EQUATIONS

A six-degree-of-freedom (6 DOF), triple-octahedron, variable-geometry truss manipulator is represented schematically in Fig. 1. The manipulator consists of

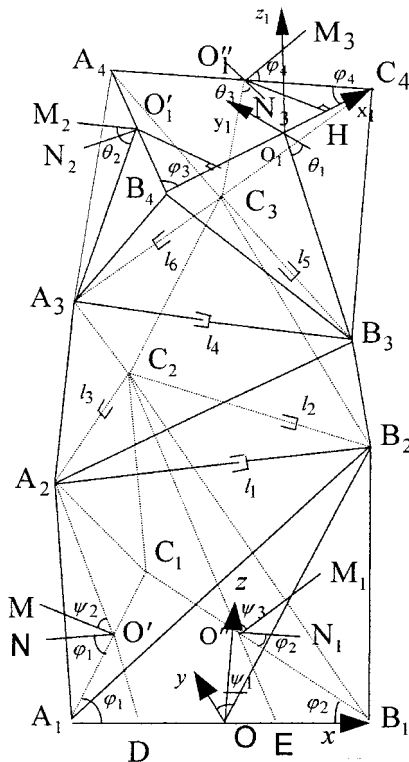


Fig. 1 A six-degree-of-freedom, triple-octahedron, variable-geometry truss manipulator

three octahedra  $A_{i+1}B_{i+1}C_{i+1} - A_iB_iC_i$  ( $i = 1, 2, 3$ ) stacked upon one another. This includes an end-effector platform  $A_4B_4C_4$ , a base platform  $A_1B_1C_1$  and two middle actuated planes in which six extensible links are located respectively. Referring to Fig. 1, the fixed coordinate system  $oxyz$  is rigidly attached to the base platform so that the  $z$  axis coincides with the normal to the base face and the  $x$  axis aligns with line  $A_1B_1$ . The moving coordinate system  $o_1x_1y_1z_1$  is attached to the top triangular face so that the  $z_1$  axis coincides with the normal to the top face and the  $x_1$  axis is aligned with line  $B_4C_4$ . Let  $\psi_1, \psi_2, \psi_3$  denote respectively the dihedral angles between planes  $A_1B_1B_2, A_1C_1A_2, B_1C_1C_2$  and plane  $A_1B_1C_1$ , and  $\theta_1, \theta_2, \theta_3$  denote respectively the dihedral angles between planes  $B_4C_4B_3, A_4B_4A_3, A_4C_4C_3$  and plane  $A_4B_4C_4$ . The lines  $B_2O, A_2O'$  and  $C_2O''$  are perpendicular to lines  $A_1B_1, A_1C_1$  and  $B_1C_1$  respectively, and lines  $B_3O_1, A_3O'_1$  and  $C_3O''_1$  are orthogonal to lines  $B_4C_4, A_4B_4$  and  $A_4C_4$  respectively.

According to the coordinate system established above, the position vectors of points  $A_2, B_2$  and  $C_2$  in the fixed coordinate system  $oxyz$  can be written as follows:

$$\begin{aligned}
 A_2 &= \begin{bmatrix} -O'A_2c\psi_2s\varphi_1 - O'D \\ O'A_2c\psi_2c\varphi_1 + O'D \\ O'A_2s\psi_2 \end{bmatrix} \\
 B_2 &= \begin{bmatrix} 0 \\ OB_2c\psi_1 \\ OB_2s\psi_1 \end{bmatrix} \\
 C_2 &= \begin{bmatrix} O''C_2c\psi_3s\varphi_2 + O''E \\ O''C_2c\psi_3c\varphi_2 + O''E \\ O''C_2s\psi_3 \end{bmatrix}
 \end{aligned} \tag{1}$$

The position vectors of points  $A_3, B_3$  and  $C_3$  in the moving coordinate system  $o_1x_1y_1z_1$  can be derived and expressed in the fixed coordinate system  $oxyz$  as follows:

$$\begin{aligned}
 \begin{bmatrix} A_3 \\ 1 \end{bmatrix} &= [T] \begin{bmatrix} -A_3O'_1c\theta_2s\varphi_3 - O'_1G \\ A_3O'_1c\theta_2c\varphi_3 + O'_1G \\ A_3O'_1s\theta_2 \\ 1 \end{bmatrix} \\
 \begin{bmatrix} B_3 \\ 1 \end{bmatrix} &= [T] \begin{bmatrix} 0 \\ O_1B_3c\theta_1 \\ O_1B_3s\theta_1 \\ 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{bmatrix} \mathbf{C}_3 \\ 1 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} \mathbf{O}_1''\mathbf{C}_3\mathbf{c}\theta_3\mathbf{s}\varphi_4 + \mathbf{O}_1''\mathbf{H} \\ \mathbf{O}_1''\mathbf{C}_3\mathbf{c}\theta_3\mathbf{c}\varphi_4 + \mathbf{O}_1''\mathbf{H} \\ \mathbf{O}_1''\mathbf{C}_3\mathbf{s}\theta_3 \\ 1 \end{bmatrix} \tag{2}$$

The constraint equations of the reverse displacement analysis problem of triple-octahedron VGTMS can be written as follows:

$$\begin{aligned} (\mathbf{B}_3 - \mathbf{B}_2)^T(\mathbf{B}_3 - \mathbf{B}_2) &= m_1^2 \\ (\mathbf{C}_3 - \mathbf{B}_2)^T(\mathbf{C}_3 - \mathbf{B}_2) &= m_2^2 \\ (\mathbf{C}_3 - \mathbf{C}_2)^T(\mathbf{C}_3 - \mathbf{C}_2) &= m_3^2 \\ (\mathbf{A}_3 - \mathbf{C}_2)^T(\mathbf{A}_3 - \mathbf{C}_2) &= m_4^2 \\ (\mathbf{A}_3 - \mathbf{A}_2)^T(\mathbf{A}_3 - \mathbf{A}_2) &= m_5^2 \\ (\mathbf{B}_3 - \mathbf{A}_2)^T(\mathbf{B}_3 - \mathbf{A}_2) &= m_6^2 \end{aligned} \tag{3}$$

where  $m_1, m_2, \dots, m_6$  are the lengths of the fixed-length links  $\mathbf{B}_2\mathbf{B}_3, \mathbf{B}_2\mathbf{C}_3, \mathbf{C}_2\mathbf{C}_3, \mathbf{C}_2\mathbf{A}_3, \mathbf{A}_2\mathbf{A}_3$  and  $\mathbf{A}_2\mathbf{B}_3$  respectively. After substitution of equations (1) and (2) and triangular identity  $\mathbf{c}\psi_i = (1 - x_i^2)/(1 + x_i^2), \mathbf{s}\psi_i = (2x_i)/(1 + x_i^2), \mathbf{c}\theta_i = (1 - y_i^2)/(1 + y_i^2), \mathbf{s}\theta_i = (2y_i)/(1 + y_i^2)$  ( $i = 1, 2, 3$ ) into equations (3) and rearrangement, the following are obtained:

$$\begin{aligned} (k_{11}y_1^2 + k_{12}y_1 + k_{13})x_1^2 + (k_{14}y_1^2 + k_{15}y_1 + k_{16})x_1 \\ + (k_{17}y_1^2 + k_{18}y_1 + k_{19}) &= 0 \\ (k_{21}y_3^2 + k_{22}y_3 + k_{23})x_1^2 + (k_{24}y_3^2 + k_{25}y_3 + k_{26})x_1 \\ + (k_{27}y_3^2 + k_{28}y_3 + k_{29}) &= 0 \\ (k_{31}y_3^2 + k_{32}y_3 + k_{33})x_3^2 + (k_{34}y_3^2 + k_{35}y_3 + k_{36})x_3 \\ + (k_{37}y_3^2 + k_{38}y_3 + k_{39}) &= 0 \\ (k_{41}y_2^2 + k_{42}y_2 + k_{43})x_3^2 + (k_{44}y_2^2 + k_{45}y_2 + k_{46})x_3 \\ + (k_{47}y_2^2 + k_{48}y_2 + k_{49}) &= 0 \\ (k_{51}y_2^2 + k_{52}y_2 + k_{53})x_2^2 + (k_{54}y_2^2 + k_{55}y_2 + k_{56})x_2 \\ + (k_{57}y_2^2 + k_{58}y_2 + k_{59}) &= 0 \end{aligned}$$

$$\begin{aligned} (k_{61}y_1^2 + k_{62}y_1 + k_{63})x_2^2 + (k_{64}y_1^2 + k_{65}y_1 + k_{66})x_2 \\ + (k_{67}y_1^2 + k_{68}y_1 + k_{69}) &= 0 \end{aligned} \tag{4}$$

### 3 ELIMINATION OF EQUATION

Equation (4) can be rewritten in the following form:

$$u_1x_1^2 + v_1x_1 + w_1 = 0 \tag{5}$$

$$u_2x_1^2 + v_2x_1 + w_2 = 0 \tag{6}$$

$$u_3x_3^2 + v_3x_3 + w_3 = 0 \tag{7}$$

$$u_4x_3^2 + v_4x_3 + w_4 = 0 \tag{8}$$

$$u_5x_2^2 + v_5x_2 + w_5 = 0 \tag{9}$$

$$u_6x_2^2 + v_6x_2 + w_6 = 0 \tag{10}$$

Multiplying equations (5) and (6) by  $x_1$ , two additional equations are obtained. The total four equations can be represented by the following matrix form:

$$\begin{bmatrix} u_1 & v_1 & w_1 & 0 \\ 0 & u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 & 0 \\ 0 & u_2 & v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 \\ x_1 \\ 1 \end{bmatrix} = 0 \tag{11}$$

The necessary and sufficient condition of existence of a non-zero solution for equation (11) is that the determinant of the coefficient matrix is equal to zero. This results in the following polynomial equation:

$$q_1y_1^4 + q_2y_1^3 + q_3y_1^2 + q_4y_1 + q_5 = 0 \tag{12}$$

where coefficients  $q_i$  ( $i = 1, 2, \dots, 5$ ) are not higher than fourth-order polynomials about  $y_3$ . Similarly, eliminating  $x_3^2$  and  $x_3$  from equations (7) and (8) and  $x_2^2$  and  $x_2$  from equations (9) and (10) respectively gives

$$a_1y_2^4 + b_1y_2^3 + c_1y_2^2 + d_1y_2 + e_1 = 0 \tag{13}$$

$$a_2y_2^4 + b_2y_2^3 + c_2y_2^2 + d_2y_2 + e_2 = 0 \tag{14}$$

where  $a_1, b_1, c_1, d_1$  and  $e_1$  are all the polynomials about  $y_3$ , the order of which is not higher than 4, and  $a_2, b_2, c_2, d_2$  and  $e_2$  are all the polynomials about  $y_1$ , the order of which is not higher than 4.

Equations (13) and (14) can be grouped into four sets. Eliminating  $y_2^4, y_2^3, y_2^2$  and  $y_2$  from each of the equations gives the following four cubic equations [10]:

$$\begin{aligned}
 (ab)y_2^3 + (ac)y_2^2 + (ad)y_2 + (ae) &= 0 \\
 (ac)y_2^3 + [(ad) + (bc)]y_2^2 + [(ae) + (bd)]y_2 + (be) &= 0 \\
 (ad)y_2^3 + [(ae) + (bd)]y_2^2 + [(be) + (cd)]y_2 + (ce) &= 0 \\
 (ae)y_2^3 + (be)y_2^2 + (ce)y_2 + (de) &= 0
 \end{aligned}
 \tag{15}$$

from which it is possible to form the following system of equations:

$$\begin{bmatrix}
 (ab) & (ac) & (ad) & (ae) \\
 (ac) & (ad) + (bc) & (ae) + (bd) & (be) \\
 (ad) & (ae) + (bd) & (be) + (cd) & (ce) \\
 (ae) & (be) & (ce) & (de)
 \end{bmatrix}
 \begin{bmatrix}
 y_2^3 \\
 y_2^2 \\
 y_2 \\
 1
 \end{bmatrix} = 0
 \tag{16}$$

where  $(ab) = a_1b_2 - a_2b_1$ , etc.

By making the determinant of the coefficient matrix equal to zero, the following equation is obtained:

$$p_1y_1^{16} + p_2y_1^{15} + p_3y_1^{14} + \dots + p_{16}y_1 + p_{17} = 0
 \tag{17}$$

where  $p_i$  ( $i = 1, 2, \dots, 17$ ) are not higher than 16th-order polynomials about  $y_3$ .

Multiplying equation (17) separately by  $y_1, y_1^2, y_1^3$  and equation (12) separately by  $y_1, y_1^2, \dots, y_1^{14}, y_1^{15}$  gives 20 equations in matrix form as follows:

$$\begin{bmatrix}
 p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & \dots & p_{16} & p_{17} & 0 & 0 & 0 \\
 0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & \dots & p_{15} & p_{16} & p_{17} & 0 & 0 \\
 0 & 0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & \dots & p_{14} & p_{15} & p_{16} & p_{17} & 0 \\
 0 & 0 & 0 & p_1 & p_2 & p_3 & p_4 & p_5 & \dots & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} \\
 q_1 & q_2 & q_3 & q_4 & q_5 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 0 & q_1 & q_2 & q_3 & q_4 & q_5 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & q_1 & q_2 & q_3 & q_4 & q_5 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & q_1 & q_2 & q_3 & q_4 & q_5 & \dots & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & q_1 & q_2 & q_3 & q_4 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & \dots & & & & & & y_1^2 \\
 & & & & & & & & \dots & & & & & & y_1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & q_1 & q_2 & q_3 & q_4 & q_5 \\
 & & & & & & & & & & & & & & 1
 \end{bmatrix}
 \begin{bmatrix}
 y_1^{19} \\
 y_1^{18} \\
 y_1^{17} \\
 \dots \\
 \dots \\
 \dots \\
 \dots \\
 \dots \\
 \dots \\
 y_1^2 \\
 y_1 \\
 1
 \end{bmatrix} = 0
 \tag{18}$$

For equation (18) to have a non-trivial solution, the determinant of the coefficient matrix is set equal to zero, and thus an output displacement equation containing only one variable  $y_3$  is obtained. This is a 128th-order algebraic equation about  $y_3$ . For each value of  $y_3$ , the corresponding  $y_1$  can be obtained from equation (18),  $y_2$  from equation (16) and  $x_1$  from (11). Similarly to the computation of  $x_1$ , the variables  $x_2$  and  $x_3$  can also be found.

As soon as  $x_i$  and  $y_i$  ( $i = 1, 2, 3$ ) are found,  $\psi_1, \psi_2, \psi_3$  and  $\theta_1, \theta_2, \theta_3$  can be evaluated from triangular formulae, and then the position vectors of points  $A_i, B_i, C_i$  ( $i = 2, 3$ ) in the  $oxyz$  coordinate system can be computed by substituting  $\psi_1, \psi_2, \psi_3$  and  $\theta_1, \theta_2, \theta_3$  into equations (1) and (2). Furthermore, the lengths of actuator members  $l_i$  ( $i = 1, 2, \dots, 6$ ) can be found.

### 4 NUMERICAL EXAMPLE

A triple-octahedron VGTM is taken as an example to explain the method. The lengths of fixed-length links, each side of the end-effector triangular platform and each side of the base triangular platform, are all 30 mm, i.e.

$$\begin{aligned}
 A_iB_i = B_iC_i = A_iC_i &= 30 \text{ mm}, & i = 1, 4 \\
 A_iA_{i+1} = B_iB_{i+1} = C_iC_{i+1} &= 30 \text{ mm}, & i = 1, 2, 3 \\
 A_iB_{i+1} = B_iC_{i+1} = C_iA_{i+1} &= 30 \text{ mm}, & i = 1, 2, 3
 \end{aligned}$$

**Table 1** Eighteen sets of real roots for equation (4)

	$x_1$	$x_2$	$x_3$	$y_1$	$y_2$	$y_3$
1	0.752	1.991	1.467	-2.936	-3.991	-0.650
2	0.258	1.994	1.497	-0.389	-3.988	-0.650
3	0.707	1.414	1.414	-0.271	-4.253	-0.627
4	0.189	2.615	1.681	-0.493	-3.202	-0.738
5	0.144	4.620	3.544	-0.707	-1.414	1.179
6	0.707	1.414	1.414	-2.708	-4.253	-7.335
7	0.144	0.477	0.347	-0.673	-1.402	-1.419
8	0.144	0.483	0.346	-0.707	-1.414	-1.414
9	1.627	0.483	3.544	-0.707	-1.414	-1.414
10	0.707	1.414	1.449	-0.271	-0.446	-0.738
11	0.707	1.414	1.414	-2.708	-0.446	-7.335
12	0.707	1.414	1.414	-0.271	-0.446	-0.627
13	1.806	0.500	3.429	-0.783	-1.456	-2.535
14	1.627	4.620	3.544	-0.707	-1.414	-1.414
15	0.144	4.620	0.346	-0.707	-1.414	-1.414
16	1.627	0.483	0.346	-0.707	-1.414	-1.414
17	1.578	4.367	0.323	-0.686	-1.853	-1.347
18	0.144	0.483	3.544	-0.707	-1.414	-1.414

The position and orientation of the end-effector are given below:

$$[\mathbf{T}] = \begin{bmatrix} 0.967 & -0.259 & 0 & 6 \\ 0.259 & 0.967 & 0 & 7 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All 128 sets of roots for equations (4) are obtained by running a program on the computer. The solutions are verified. Eighteen sets of real roots are listed in Table 1.

**5 CONCLUSIONS**

In this paper, closed-form solutions for the inverse kinematic analysis of a triple-octahedron, variable-geometry truss manipulator are presented for the first time. A 128th-degree algebraic equation in one unknown is derived. A numerical example is tested. The results show the method is simple, effective and accurate. In the experimental computation, no extraneous roots are found.

**ACKNOWLEDGEMENT**

This research work was supported by the China National Science Foundation under grants NSFC 59475002 and 59275157.

**REFERENCES**

- Arun, V., Reinholtz, C. F. and Watson, L. T.** Enumeration and analysis of variable geometry truss manipulators. In Proceedings of ASME Mechanisms Conference, 1990, pp. 93–98 (American Society of Mechanical Engineers, New York).
- Miura, K., Furuya, H. and Suzuki, K.** Variable geometry truss and its application to deployable truss and space crane arm. *Acta Astronaut.*, 1985, **12**(7).
- Reinholtz, C. F. and Gokhale, D.** Design and analysis of variable geometry truss robots. In Proceedings of 10th Applied Mechanism Conference, December 1987.
- Padmanabhan, B., Arun, V. and Reinholtz, C. F.** Closed-form inverse kinematic analysis of variable geometry truss manipulators. In Proceedings of ASME Mechanisms Conference, 1990, pp. 99–105 (American Society of Mechanical Engineers, New York).
- Jain, S. and Kramer, S. N.** Forward and inverse kinematic solution of variable geometry truss robot based on *N*-celled tetrahedron–tetrahedron truss. *Trans. ASME, J. Mech. Des.*, 1990, **112**(1), 16–22.
- Subramaniam, M. and Kramer, S. N.** The inversion kinematic solution of the tetrahedron based variable geometry truss manipulators. *Trans. ASME, J. Mech. Des.*, 1992, **114**, 433–437.
- Arun, V., Reinholtz, C. F. and Watson, L. T.** Application of new homotopy continuation techniques to variable geometry trusses. *Trans. ASME, J. Mech. Des.*, 1992, **114**(3).
- Griffs, M. and Duffy, J.** A forward displacement analysis of a class of Stewart platform. *J. Robotic Syst.*, December 1989, **6**(6).
- Xu, L. J. and Yang, S. X.** Reverse displacement analysis of triple-octahedron variable geometry truss manipulators. In Proceedings of International Conference on *Mechanical Transmissions and Mechanisms (MTM'97)*, Tianjin, China, 1997.
- Duffy, J. and Rooney, J.** Displacement analysis of spatial six-link, 5R-C mechanisms. *Trans. ASME, J. Appl. Mechanics*, 1974, **41**(3), 1–9.