BALANCING ON INCLINED SURFACE

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Summary

Balancing systems placed on inclined surface are examined. The stability analysis of the inverted pendulum model is accomplished and the stability chart in the plane of the control parameters is constructed. The influence of the sampling delay appearing in digital control and the inclination of the slope is revealed. Human beings are able to balance themselves in spite of the inability to detect the exact vertical line. The mechanical model of the human balancing organ is also investigated. The stability analysis is implemented and the stability chart is drawn again. The effect of detecting the angle between the pendulum and the surface instead of the angle of the inclination is discussed.

1 INTRODUCTION

Balancing is a basic example of stabilizing unstable equilibria of mechanical systems. The main application areas in engineering disciplines are biomechanics and robotics [3,7]. The simplest model of balancing is that of the inverted pendulum [1,2,4,5,6]. The control system needs an absolute reference line as vertical direction. The control force points in the direction of the inclination, so it has vertical component if the system is considered on an inclined surface. In other words, the gravitational force has component in the direction of the inclination, so the control force must keep the pendulum on the surface. These problems have an influence on the stability conditions. The digital control of the inverted pendulum on a slope is discussed in chapter 2.

Healthy human beings are able to balance themselves successfully during standing and walking on a surface with unknown inclination or even without knowing where the exact verical direction is. The human balancing organ called labyrinth is responsible for successful balancing. The examined model is the mechanical model of the labyrinth [10,11]. The computer control becomes successful without knowing the exact vertical direction, because only relative angular velocities determine the control force. However, the control force includes a part depending on the angle of the inclination. The angle between the pendulum and the surface can be detected, but the angle of the inclination cannot be determined exactly if the position of the pendulum is not exactly vertical. This effect also changes the stability conditions. The digital control of the artificial labyrinth on a slope is accomplished in chapter 3.

2 THE INVERTED PENDULUM ON INCLINED SURFACE

The well-known model of balancing is that of the inverted pendulum. Let us put this model on an inclined surface as it is depicted on Figure 1. The system has 2 degrees of freedom described by the general coordinates, the horizontal displacement x and the angle of the pendulum φ. A geometrical constraint determines the vertical displacement y which is the function of the horizontal displacement.

$$y = h(x)\,, \quad H = \frac{dh(x)}{dx} = \tan\alpha\,. \tag{1}$$

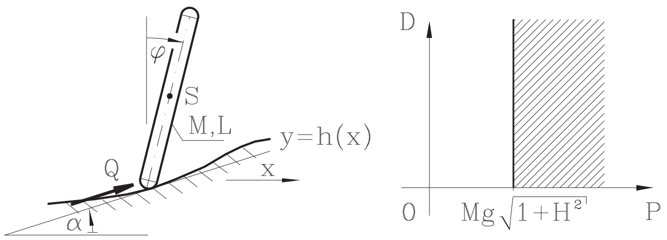

Figure 1: *The inverted pendulum on inclined surface and its stability map*

After the algebraic elimination of $\ddot{x}$, the linearized equation of motion regarding to φ has this form:

$$\ddot{\varphi} = \frac{6g(1+H^2)}{L(1+4H^2)}\varphi + \frac{6g}{L(1+4H^2)}H - \frac{6}{ML(1+4H^2)}Q\,, \tag{2}$$

where supposing τ sampling delay in the system the control force Q is given by

$$Q\,(t) = \left(P\varphi\,(t-\tau) + D\dot{\varphi}\,(t-\tau) + \frac{MgH\,(x\,(t-\tau))}{\sqrt{1+H\,(x\,(t-\tau))^2}}\right)\sqrt{1+H\,(x\,(t))^2}\,. \tag{3}$$

The control force must keep the pendulum on the surface, so it does not vanish even if the angle and the angular velocity of the pendulum is 0. A part proportional to the inclination of the slope appears in the control force. Let us assume that the inclination changes so slow that $H\,(x\,(t)) \approx H\,(x\,(t-\tau))$. Then the mentioned part in the control force eliminates the nonlinear part of the equation of motion, so it can be written in the following form:

$$\ddot{\varphi} + \frac{6\sqrt{1+H^2}D}{ML(1+4H^2)}\dot{\varphi} + \left(\frac{6\sqrt{1+H^2}P}{ML(1+4H^2)} - \frac{6g(1+H^2)}{L(1+4H^2)}\right)\varphi = 0\,. \tag{4}$$

The stability analysis is carried out by the Routh-Hurwitz criterion after discretization and applying the Moebius-transformation. If $\tau = 0$ then the $\varphi \equiv 0$ trivial solution of (4) is asymptotically stable if and only if

$$P > P_0 = Mg\sqrt{1+H^2}\,, \quad D > 0\,. \tag{5}$$

The stability conditions for smooth horizontal surface can be obtained substituting $H = 0$ in (5). This result is equivalent to that published in [8]. Obviously, the border of the stability domain moves toward greater values of P as H increases.

If $\tau > 0$, then the trivial solution of (4) is asymptotically stable if and only if

$$P > P_0\,, \quad H_2 > 0\,, \tag{6}$$

where H_2 is the maximum-sized Hurwitz-determinant.

The stability charts without time delay can be seen in Figure 1 and for different values of the inclination of the slope with time delay are given in Figure 2(a). A simple stick is considered with mass $M = 0.1$ [kg] and length $L = 0.5$ [m]. Both the straight line and the parabola bounding the stability domain move toward greater values of P as H increases.

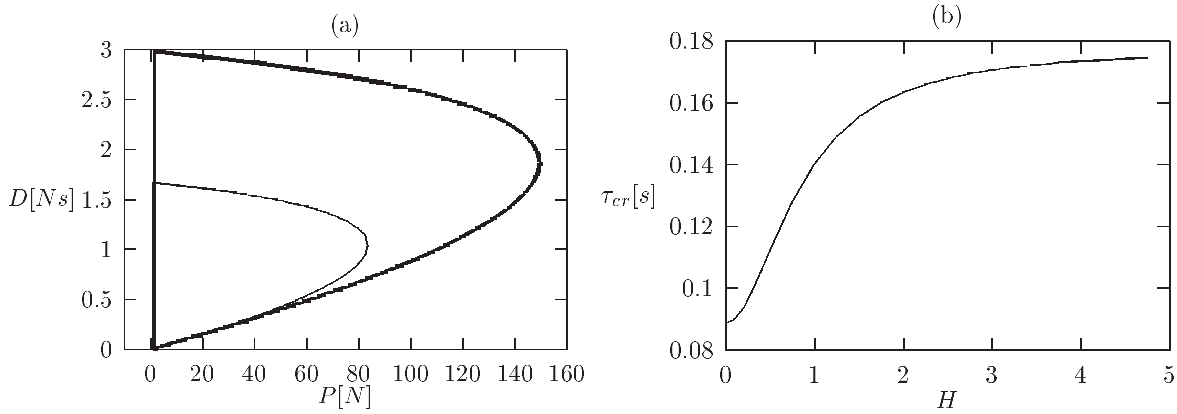

Figure 2: *(a) The stability charts,* $H = 0$ *and* $\tau = 5[ms]$ *(thin line),* $H = 0.5$ *and* $\tau = 5[ms]$ *(thick line), (b) The critical time delay vs. the inclination of the slope*

The stability domain disappears at a certain critical value of the time delay. It has the following form:

$$\tau_{cr} = \sqrt{\frac{L(1+4H^2)}{6g(1+H^2)}} \ln \frac{3+\sqrt{5}}{2} \tag{7}$$

This critical value versus the inclination of the slope is drawn in Figure 2(b). It seems that the balancing is easier on an inclined surface but the necessary control force tends to infinity as $H \to \infty$.

3 USE OF THE ARTIFICIAL LABYRINTH ON INCLINED SURFACE

The mechanical model of the labyrinth has been constructed in [10,11] as it is sketched in Figure 3 and the linearized equations of motion have been determined.

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\dot{\mathbf{y}} + \mathbf{S}\mathbf{y} = \mathbf{b}\,, \quad \mathbf{y} = (\varphi \quad \gamma \quad \beta)^T \,. \tag{8}$$

The only difference is present in the control force Q implied in $\mathbf{b}$ in case of the artificial labyrinth putting on a slope.

3.1 The problem of sensing the angle of the inclination

Computer control is successful without the knowledge of the absolute vertical direction, but the control force should be proportional to the angle α of the inclination. Only the angle $\alpha + \varphi$ between the pendulum and the surface can be detected, therefore the control force is proportional to this angle. After some algebraic calculations it can be obtained as it follows:

$$Q = \left(P\left(\dot{\varphi} + \dot{\gamma}\right) + D\left(\dot{\varphi} - \dot{\beta}\right)\right)\sqrt{1+H^2} + M_\Sigma g H + Q^*\,, \tag{9}$$

where

$$M_\Sigma = M + m + m_e\,, \quad Q^* = M_\Sigma g \varphi\,. \tag{10}$$

Neglecting the delay effect the stability analysis is implemented again using the Routh-Hurwitz criterion. The trivial solution of (8) is asymptotically stable if and only if the coefficient of the constant part a_0 in the characteristic equation and the maximum-sized Hurwitz-determinant H_4 is positive. The stability charts are shown in Figure 3. The border of the stability domain moves in the direction of positive P and negative D as the inclination of the slope increases.

A surprising result is depicted in Figure 4(a). The size of the stability domain is greater if the relative angle between the pendulum and the surface is detected. An interesting property of the control force explains this fact. Q^* is proportional to the angle of the pendulum, and the gain would be 0 if the angle of the inclination were detected exactly, while it is $M_\Sigma g$ if the mentioned relative angle is detected. The control force has a part proportional to the angle of the pendulum, which improves stability in the control.

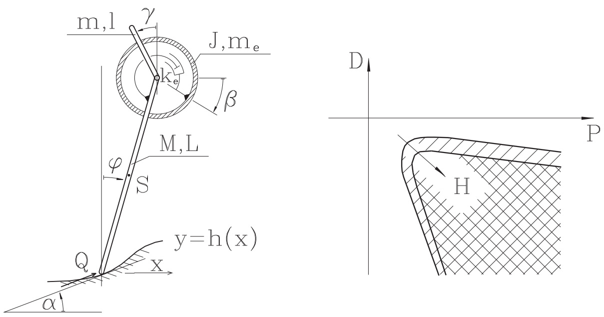

Figure 3: *The artificial labyrinth on inclined surface and its stability map with a parameter*

3.2 The delay effect

Considering the delay effect in the system the discretization and the Moebius-transformation have to be applied again before the usage of the Routh-Hurwitz criterion. The discrete equations of motion is given by

$$\mathbf{\tilde{y}_{n+1}} = \mathbf{A_d}\mathbf{\tilde{y}_n} + \mathbf{B_d}\mathbf{\tilde{u}_n}\,, \quad \mathbf{\tilde{u}_{n+1}} = \mathbf{D}\mathbf{\tilde{y}_n}\,, \tag{11}$$

where $\mathbf{A_d}, \mathbf{B_d}$ and $\mathbf{D}$ is determined by $\mathbf{M}, \mathbf{K}, \mathbf{S}$ and $\mathbf{b}$, but they are not presented here algebraically due to their complicated forms.

The trivial solution of (11) is asymptotically stable if and only if the coefficient of the constant part a_0 in the characteristic equation and the maximum-sized Hurwitz-determinant H_6 is positive. The stability charts for different values of the time delay are constructed in Figure 4(b).

4 CONCLUSIONS

The digital control of balancing systems on inclined surface has been examined. The inclination makes the minimum of the proportional gain of the control force which is necessary for successful balancing greater, but the maximum of that is also getting greater and the critical sampling delay increases as the incline of the slope increases. The balancing does still not become easier, because the control force ought to increase without limit.

Human balancing is successful without knowing where the exact vertical direction is even on inclined surface. Therefore its mechanical model is placed on inclined surface and the control system detects relative angles and velocities. Detecting the angle between the pendulum and the surface a part proportional to the angle of the pendulum is occured in the control force which helps the control system, therefore the stability domain is a bit extended in the direction of smaller gains.

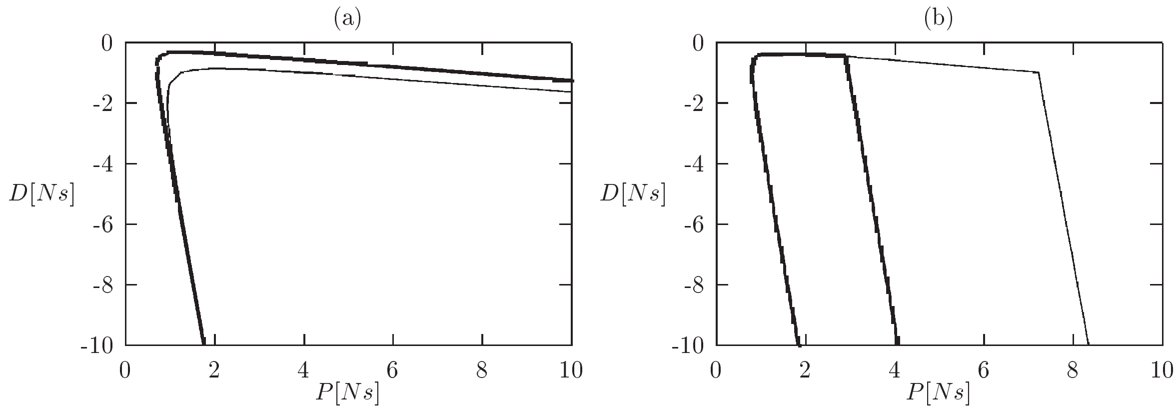

Figure 4: *The stability charts, (a)* α *is detected (thin line),* $\alpha+\varphi$ *is detected (thick line), (b)* $\tau = 0.4[ms]$ *(thin line) and for* $\tau = 1[ms]$ *(thick line)*

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