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BACKLASH IN MACHINES STABILIZED BY CONTROL FORCE

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Summary

There are a lot of problems when balancing an unstable equilibrium of mechanical systems stabilized by control force. Time delay, driving through elastic belt, backlash at the driving-wheel of the engine destabilize mechanical systems. Effects of these factors will be examined in this paper.

1 INTRODUCTION

Unstable equilibria of mechanical systems often have to be stabilized by control force in mechanical engineering. A typical example of this is the balancing of walking and standing robots. The simplest model of balancing is the inverted pendulum. Stabilization of the inverted pendulum is a challenging basic example, so a long series of publications has appeared in specialized literature in the last forty year (see [1,2,3,4,5]) either about its theorotical or experimental aspects.

The system which executes controlling is considered by a cart. The inverted pendulum and the motor displaying the control force is placed on this cart and the motor drives one of the wheels of the cart through a driving-belt. Controlling is executed by a computer which is situated outside this cart. There are two general factors which influences the stability conditions: sampling delay and stiffness of the driving-belt. Increasing time delay and elasticity of the driving-belt tends to destabilize the examined system.

Considering the backlash at the driving-wheel of the motor, the correct way of balancing the upper position of the pendulum is impossible. The pendulum swings with little amplitude around its equilibrium. The stability domain in the plane of the control parameters did not change, but an unstable zone had appeared in phase-diagram. The trivial solution is an unstable equilibrium, but there is a stable closed orbit which determines the movement of the pendulum around its equilibrium.

In the subsequent chapters the stability charts in the plane of the control parameters are constructed in case of simple inverted pendulum first, then in case of driving system. The influence of the elasticity of the driving-belt is considered and the stable closed orbit in phase-diagram is investigated.

2 THE INVERTED PENDULUM

The simplest possible model of balancing is the inverted pendulum shown in Figure 1 [6,8]. The system has 2 degrees of freedom described by the general coordinates, x and φ. The angle φ and the angular velocity $\dot{\varphi}$ is detected and the control force Q is determined by them in a way that the upper position of the pendulum should be asymptotically stable.

The nonlinear equations of motion assume the form

$$\begin{pmatrix} m & \frac{1}{2}ml\cos\varphi \\ \frac{1}{2}ml\cos\varphi & \frac{1}{3}ml^2 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} - \begin{pmatrix} \frac{1}{2}ml\dot{\varphi}^2\sin\varphi \\ \frac{1}{2}mgl\sin\varphi \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix} . \qquad (1)$$

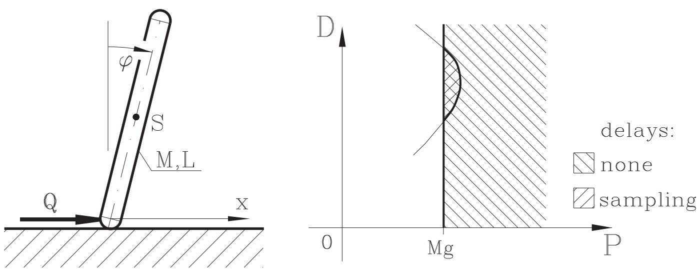

Figure 1: *The simple inverted pendulum model and its stability map*

After the algebraic elimination of $\ddot{x}$, the linearized equation of motion assumes the form:

$$\ddot{\varphi} - \frac{6g}{l}\varphi = -\frac{6}{ml}Q\,. \tag{2}$$

The control force Q is determined as follows:

$$Q\left(t\right) \equiv P\varphi\left(t_j - \tau\right) + D\dot{\varphi}\left(t_j - \tau\right)\,, \quad t \in \left[j\tau, \left(j+1\right)\tau\right)\,, \tag{3}$$

where τ is the sampling delay.

The stability analysis can be carried out by the Routh-Hurwitz criterion. If $\tau = 0$, then the trivial solution of (2) is asymptotically stable if and only if $P > mg$ and $D > 0$.

If $\tau > 0$, then the trivial solution of (2) is asymptotically stable if and only if $P > mg$ and $H_2 > 0$, where H_2 is the maximum-sized Hurwitz-determinant, we obtain after the so-called Moebius-transformation. These stability conditions are represented in Figure 1.

There always exists parameters P, D such as the trivial solution of (2) is asymptotically stable if $\tau < \tau_{cr}$, and it is unstable for any P, D if $\tau > \tau_{cr}$. This value can be determined from applying equations in the stability conditions:

$$\tau_{cr} = \sqrt{\frac{l}{6g}}\ln\frac{3+\sqrt{5}}{2}\,. \tag{4}$$

Two complex conjugate roots turn up in the right half of the complex plane, and this refers to a Hopf bifurcation resulting periodic motion around the desired equilibrium. The oscillation frequency is the imaginary part of the characteristic root at the loss of stability.

3 INVERTED PENDULUM ON A CART

In the purpose of describing a realistic balancing system, the inverted pendulum is placed on a cart as it can be seen in Figure 2 [8]. The motor drives one of the wheels of this cart through a driving-belt with stiffness s. The system has 3 degrees of freedom described by the general coordinates, x, φ and ψ. The angle φ of the pendulum and the displacement x of the cart are detected together with its derivatives.

Backlash appears in the system as a nonlinear spring characteristic. Assume the alongation of the spring is Δ. The force in spring is the function of Δ:

$$R_r = \begin{cases} s\left(\Delta + r_0\right) & \Delta \leq -r_0 \\ 0 & \left|\Delta\right| < r_0 \\ s\left(\Delta - r_0\right) & \Delta \geq r_0 \end{cases}\,, \tag{5}$$

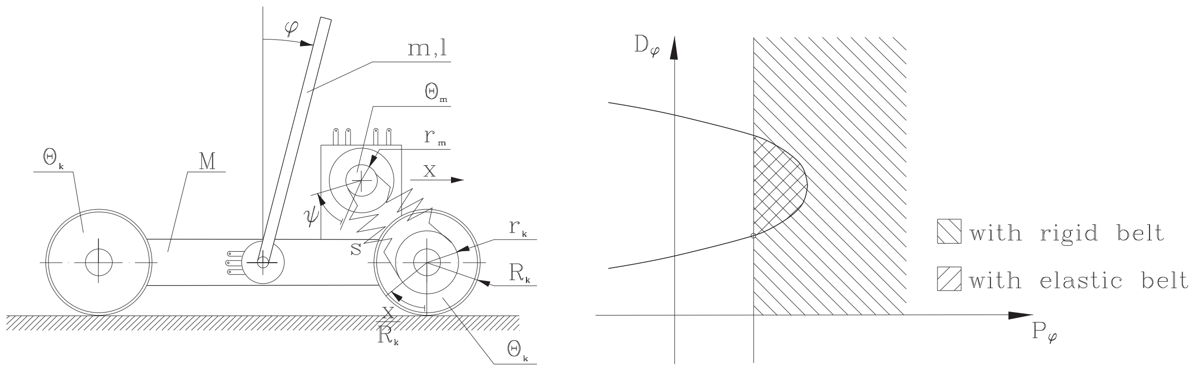

Figure 2: *The inverted pendulum on a cart and its stability map*

where r_0 is the value of backlash.

The control force is determined by the motor characteristic. The driving-moment is linearly proportional to the voltage of the motor and inversely proportional to the angular velocity:

$$\begin{aligned} M_m &= LU_m - K\dot{\psi}\,, \\ U_m &= P_\varphi \varphi + D_\varphi \dot{\varphi} + P_x x + D_x \dot{x}\,. \end{aligned} \tag{6}$$

3.1 Driving through rigid belt

If the belt is ideally rigid, then x determines ψ evidently. The system is reduced to a system with 2 degrees of freedom. The linearized equations of motion assume the form:

$$\begin{pmatrix} m + M + \frac{1}{2} m_m \frac{r_k^2}{R_k^2} & \frac{1}{2} ml \\ \frac{1}{2} ml & \frac{1}{3} ml^2 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} mgl \end{pmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}, \tag{7}$$

where the control force has the form

$$Q = \frac{r_k}{r_m R_k} L \left(P_\varphi \varphi + D_\varphi \dot{\varphi} + P_x x + D_x \dot{x} \right) - K \frac{r_k^2}{r_m^2 R_k^2} \dot{x}\,. \tag{8}$$

The system can be stabilized if the displacement of the cart is not detected and the differential gain of the cart eliminates the damping of the motor. Then the control force is simplified to:

$$Q = \frac{r_k}{r_m R_k} L \left(P_\varphi \varphi + D_\varphi \dot{\varphi} \right)\,. \tag{9}$$

The trivial solution of (7) is asymptotically stable if and only if

$$P_\varphi > \frac{1}{L} \left[(m + M)\, g \frac{r_m R_k}{r_k} + \frac{1}{2} m_m g \frac{r_m r_k}{R_k} \right], \quad D_\varphi > 0\,.$$

The stability domain is shown in Figure 2.

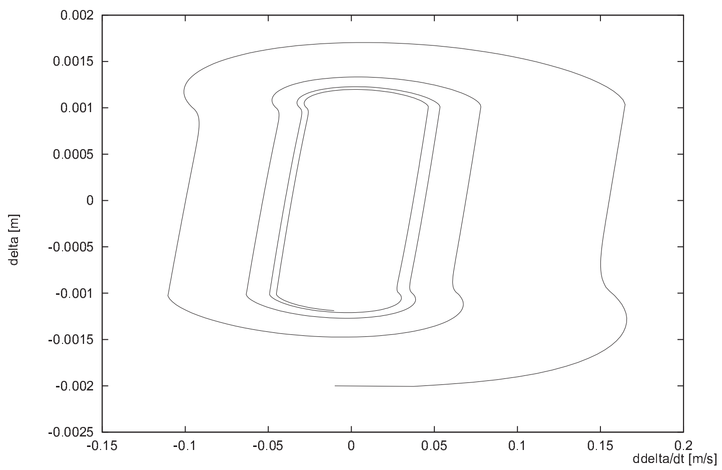

Figure 3: *Phase diagram on $\dot{\Delta} - \Delta$ plane*

3.2 Driving through elastic belt

If the driving-belt is elastic, then the equations of motion of the system with 3 degrees of freedom assume the form:

$$\begin{pmatrix} m+M & \frac{1}{2}ml & 0 \\ \frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ 0 & 0 & \frac{1}{2}m_m r_m^2 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \\ \ddot{\psi} \end{pmatrix} + \begin{pmatrix} s\frac{r_k}{R_k} & 0 & -s\frac{r_m r_k}{R_k} \\ 0 & -\frac{1}{2}mgl & 0 \\ -s\frac{r_m r_k}{R_k} & 0 & s r_m^2 \end{pmatrix} \begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Q \end{pmatrix} . \tag{10}$$

The trivial solution of this system is asymptotically stable if and only if

$$s > \frac{3\,(m+M)\,m_m g}{\left(m + 4M + 2m_m \frac{r_k^2}{R_k^2}\right) l}\,, \qquad P_\varphi > \frac{1}{L}\left[\left(m + M + \frac{1}{2}m_m \frac{r_k^2}{R_k^2}\right) g \frac{r_m R_k}{r_k}\right], \qquad H_3 > 0\,.$$

The stability domain constructed according to the former is shown in Figure 2.

3.3 Backlash in the elastic belt

Considering the backlash in the system the equations of motion will change according to the nonlinear spring characteristic that is caused by $r_0 \neq 0$. New constant expressions appear that means shifting of the solutions. The stability domain does not change but it is valid only if $|\Delta| > r_0$. Otherwise, the system is just in backlash, so it cannot be stabilized, because control force is not displayed in this little domain. The system is reduced to a system with 2 degrees of freedom, if a new general coordinate is introduced. This is the alongation of the spring:

$$\Delta = r_m \psi - \frac{r_k}{R_k} x\,. \tag{11}$$

Applying this new general coordinate the linear equations of motion assume the form:

$$\begin{pmatrix} \frac{(m+M)m_m r_m}{2} & -\frac{m m_m l r_m r_k}{4R_k} \\ 0 & \frac{ml^2}{3} - \frac{m^2 l^2}{4(m+M)} \end{pmatrix} \begin{pmatrix} \ddot{\Delta} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} (m+M)\,s r_m + \frac{m_m r_m r_k^2}{2R_k^2} s & 0 \\ \frac{mlr_k}{2(m+M)R_k} s & -\frac{mgl}{2} \end{pmatrix} \begin{pmatrix} \Delta \\ \varphi \end{pmatrix} =$$

$$= \begin{pmatrix} (m+M)\,Q \mp (m+M)\,s r_m r_0 \mp \frac{m_m r_m r_k^2}{2R_k^2} s r_0 \\ \mp \frac{mlr_k r_0}{2(m+M)R_k} s \end{pmatrix} . \tag{12}$$

Transform this system of equations to four first order differential equations. Roots of the characteristic equation are eigenvalues of the coefficient matrix of this system. There are two real and two complex conjugate roots. If P_φ and D_φ are choosen from the stability domain, then the real parts of all eigenvalues are negative. Trajectories form stable focus around the $\left(\varphi, \dot{\varphi}, \Delta, \dot{\Delta}\right) = (0, 0, \pm r_0, 0)$ equilibria.

If the system is just in backlash, then control force is not displayed, the stiffness of driving-belt is 0. Roots of the characteristic equation are positive and negative reals in this case. Trajectories form saddle around the $(0, 0, 0, 0)$ equilibrium.

Three closed orbits can be found after the examination of the phase plane [7]. There are a stable and two unstable limit cycles. Figure 3 shows a trajectory on the $\dot{\Delta} - \Delta$ plane. This trajectory spirals from outside to the stable limit cycle. The physical meaning of this result is the oscillation of the stick around the $(0, 0, 0, 0)$ equilibrium.

There are unstable limit cycles around the $(0, 0, \pm r_0, 0)$ equilibria. The physical meaning of this result is that the control force does not push the stick further than the vertical line and the stick oscillates with less and less amplitude on one side of the vertical position, if the initial conditions are inside the unstable limit cycle. This is a very little domain, it is difficult to detect it.

4 CONCLUSIONS

It is known that increasing time delay tends to destabilize dynamical systems. However, this is not the only problem in balancing systems. Increasing elasticity of the driving-belt causes instability, and backlash at the driving-wheel causes oscillation of the stick around the equilibrium. Backlash behaves as a spatial delay.

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