PERIODIC RESPONSES OF A CONTROLLED BALANCING SYSTEM

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Introduction

Dynamical systems with piecewise linear components in the equations of motion occur frequently in practice. Gear pairs with backlash, impact dampers, moving parts with dry friction and adjacent structures during earthquakes are modelled by systems with piecewise linear damping, stiffness or restoring force [1]-[4]. Analytical solution techniques which are applicable to weakly nonlinear equations are not suitable for these investigations due to the strongly nonlinear nature of the governing equations of these systems, but there exist analytical methods to determine periodic responses [5]-[8].

Control is often added to such systems. Typical cases are when unstable equilibra of mechanical systems have to be stabilized. This problem arises at the stick balancing on a cart, where the cart is driven by a motor through a teeth-belt [9]-[11]. The upper equilibrium of the stick is unstable and backlash appearing at the driving makes this system piecewise linear. Stability analysis of the system without backlash is discussed and the stability chart is constructed in previous works [11]. When backlash is present, periodic motions occur in that parameter domain where stable equilibrium was obtained by using the Routh-Hurwitz criterion without backlash. In the present paper, periodic responses are determined by applying the harmonic balance method [5] and results are compared with those obtained in [12]-[13] using the continuation method [14].

The pendulum-cart model

The above mentioned balancing system is given in Figure 1 [10]-[13]. The system has 2 degrees of freedom described by the angle φ of the pendulum and the alongation $\Delta = r_m\psi - \frac{r_w}{R_w}x$ of the belt, where r_m [m] is the radius of the motor axle, r_w [m] and R_w

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[m] are radii of the wheel, ψ is the angle of the motor axle and x [m] is the displacement of the cart. The velocity $\dot{x}$ of the cart and the angle φ of the pendulum with its derivative are measured. The differential gain based on the cart velocity may eliminate the damping K [Nms] of the motor, so the control force can be determined in the simplified form

$$Q = P\varphi + D\dot{\varphi}\,, \tag{1}$$

where P and D are the coefficients of the PD regulator. The linearized equations of motion have the form

$$\begin{pmatrix} \frac{(m+M)m_m r_m}{2} & -\frac{m m_m l r_m r_w}{4R_w} \\ 0 & \frac{ml^2}{3} - \frac{m^2 l^2}{4(m+M)} \end{pmatrix} \begin{pmatrix} \ddot{\Delta} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} (m+M)\frac{K}{r_m} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\Delta} \\ \dot{\varphi} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -\frac{mgl}{2} \end{pmatrix} \begin{pmatrix} \Delta \\ \varphi \end{pmatrix} + \begin{pmatrix} (m+M)\,r_m + \frac{m_m r_m r_w^2}{2R_w^2} \\ \frac{mlr_w}{2(m+M)R_w} \end{pmatrix} R_s = \begin{pmatrix} (m+M)\,Q \\ 0 \end{pmatrix}, \tag{2}$$

where $R_s = s\Delta$ is the force in the elastic belt, s [N/m] is the spring stifness, m [kg] and m_m [kg] are masses of the pendulum and the motor, M [kg] is sum of the mass of the cart, the motor and the reduced mass of the inertia of the wheels, l [m] is the length of the pendulum and g [m/s^2] is the gravitational acceleration.

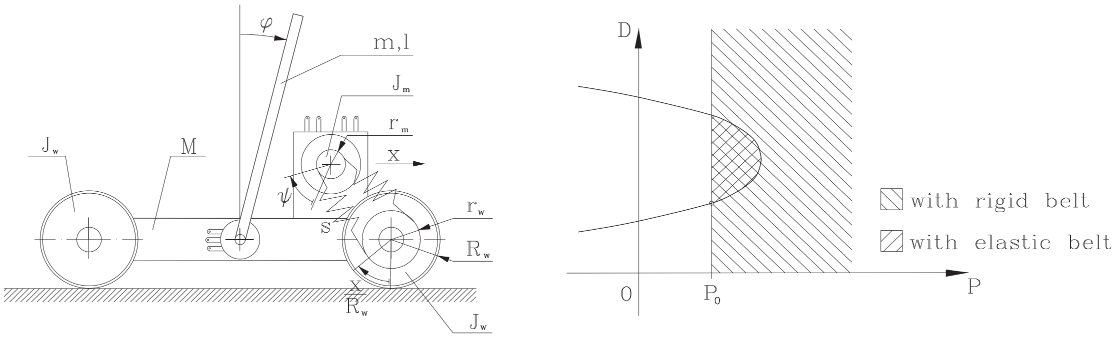

Fig.1. The inverted pendulum-cart model and its stability map

The stability analysis was carried out by the Routh-Hurwitz criterion [11]. The trivial solution of (2) is asymptotically stable if and only if

$$P > P_0 = \left(m + M + \frac{1}{2} m_m \frac{r_w^2}{R_w^2} \right) g \frac{r_m R_w}{r_w} \quad \text{and} \quad H_3 > 0\,, \tag{3}$$

where H_3 is the maximum sized Hurwitz-determinant, not presented here algebraically. The stability chart was constructed as it is shown in Figure 1.

Application of the harmonic balance method

Backlash occurs at the contact of the driving belt and the axle of the motor. The control force is not transmitted in a tiny zone, which means that the spring characteristic is nonlinear. The force in the belt is the function of the alongation Δ

$$R_s = \begin{cases} s\,(\Delta + f) & \Delta \le -f \\ 0 & |\Delta| < f \\ s\,(\Delta - f) & \Delta \ge f \end{cases}\,, \tag{4}$$

where f is the value of backlash. This function is shown in Figure 2(a).

In the followings, the general approach to the use of the harmonic balance method for piecewise linear systems, proposed in [5], is used. Let us decompose the above function to linear and saturation curves [5]. The characteristic is symmetric, therefore it is written only in the domain of positive alongation

$$R_s = s\Delta - \begin{cases} s\Delta & \Delta < f \\ sf & \Delta \geq f \end{cases} , \tag{5}$$

as it is given in Figure 2(b). Periodic solutions are looked for, so the alongation is considered in the form of $\Delta = B\sin(\omega t)$ and since the characteristic is monovalent and symmetric, the force in the belt is simplified to the following form

$$R_s = b\Delta + r_3 B \sin(3\omega t) + r_5 B \sin(5\omega t) + \ldots , \tag{6}$$

where

$$b = s\left(1 - k\left(\frac{B}{f}\right)\right) , \quad k\left(\frac{B}{f}\right) = \frac{2}{\pi}\left(\arcsin\frac{f}{B} + \frac{f}{B}\sqrt{1 - \frac{f^2}{B^2}}\right) , \tag{7}$$

$$r_3 = -s\kappa_{11}\left(\frac{B}{f}\right) , \quad r_5 = -s\kappa_{12}\left(\frac{B}{f}\right) , \quad \ldots \tag{8}$$

$$\kappa_{1m}\left(\frac{B}{f}\right) = \frac{1}{\pi}\left(\frac{\sin(2m\varphi_1)}{m} - \frac{\sin(2(m+1)\varphi_1)}{m+1} + \frac{4}{2m+1}\sin\varphi_1 \cos(2(m+1)\varphi_1)\right)$$

$$m = 1, 2, \ldots \quad \varphi_1 = \arcsin\left(\frac{f}{B}\right) . \tag{9}$$

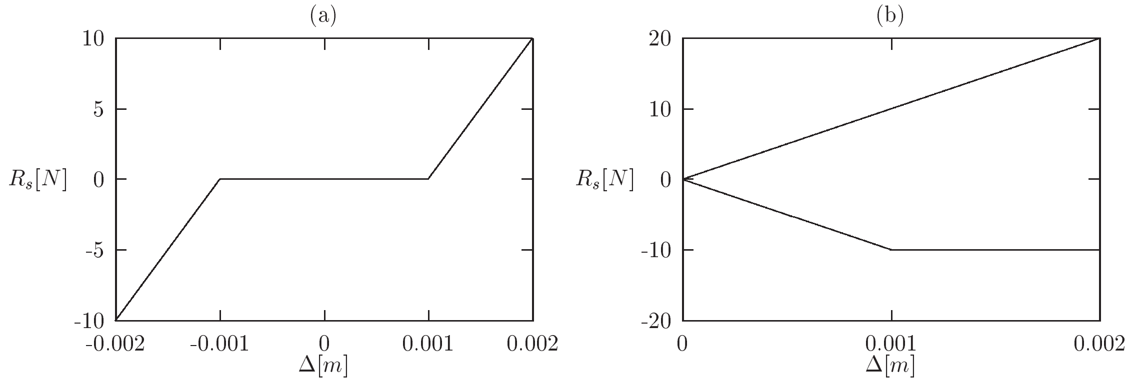

Fig.2. (a) The spring characteristic and (b) its decomposition

Let us neglect the subharmonics at the first approximation, then, $R_s = b\Delta$ substitution is applied in equation (2). Periodic solution appears if the characteristic equation has a pair of pure imaginary roots. Let λ be the characteristic root. Substituting $\lambda = i\omega$ in the characteristic equation, then separating the real and the imaginary part, two equations are obtained. Substituting equation (7) and the values of the parameters describing the system, 4 unknowns are included, the amplitude B and the angular frequency ω of the oscillation as well as the control parameters P and D. If one of the control parameters is fixed, then the angular frequency and the other control parameter are obtained as the function of the amplitude of the oscillations. Thus, the amplitude and the frequency of the periodic solution are obtained for every pair of the control parameters for which periodic solution exists.

Results

Computations are accomplished for values of the parameters described for a realised pendulum-cart system and which is given in [13] (see Appendix). The proportional gain and the angular frequency are shown by the continuous curves in Figure 3(a) and 3(b) as the function of the amplitude, if the differential gain is fixed, $D = 2$ [Nms]. Periodic solution with the smallest amplitude is obtained for $B = 1.0087$ [mm], its angular frequency is $\omega = 0.0705$ [1/s] and the corresponding proportional gain is $P = 0.1986$ [Nm], which is P_0 given in equation (3). The line bordering the stability domain in Figure 1 is obtained for this value. The straight line indicated in Figure 3(a) corresponds to the value of the proportional gain which is the P coordinate of the parabola bordering the stability domain for $D = 2$ [Nms].

The differential gain and the angular frequency are given by the continuous curves in Figure 3(c) and 3(d) as the function of the amplitude, if the proportional gain is fixed, $P = 20$ [Nm]. The straight lines in Figure 3(c) correspond the values of the differential gain which are the D coordinates of the parabola bordering the stability domain for $P = 20$ [Nm].

The dotted lines in Figure 3 are the results computed by using the continuation method and presented in [12]. Results obtained by the two method coincide well apart from a small region at the border of the stability domain where periodic solution does not exist according to the continuation method.

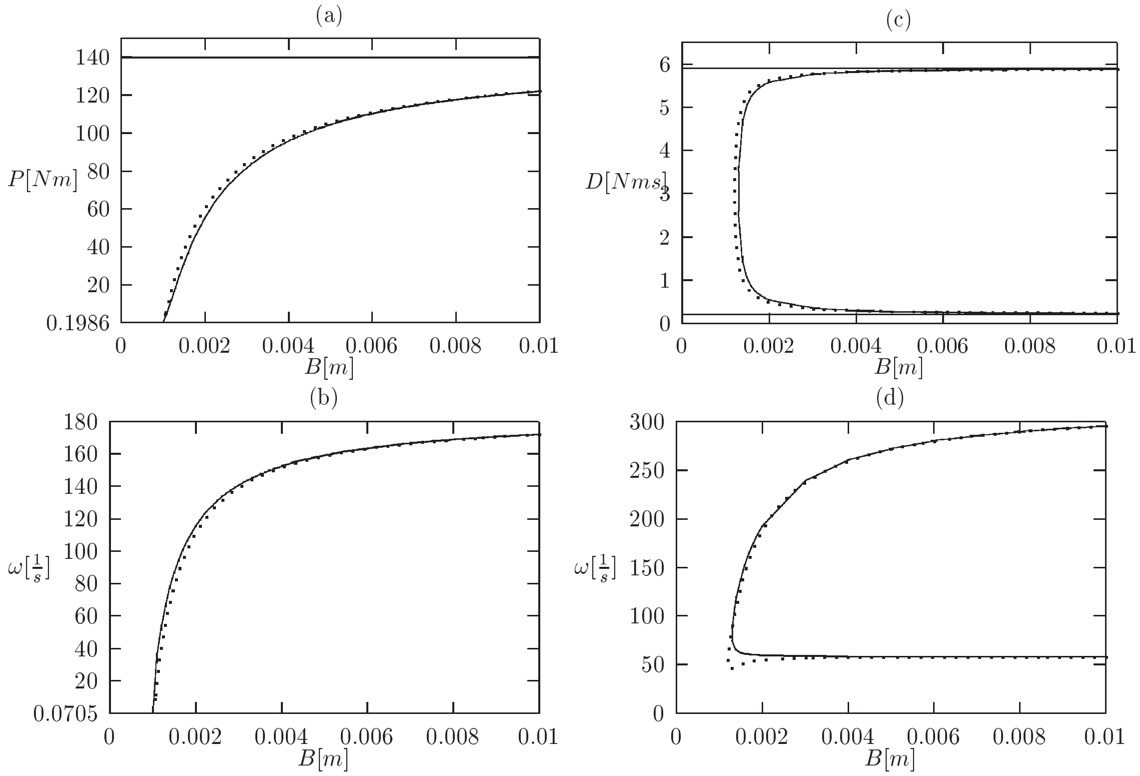

Fig.3. (a) $P - B$ function and (b) $\omega - B$ function for $D = 2$ [Nms],
(c) $D - B$ function and (d) $\omega - B$ function for $P = 20$ [Nm]

The effect of the subharmonics is also examined. The coefficient of the harmonic b, the first and the second nonzero subharmonics r_3 and r_5 as the functions of the amplitude are computed from equations (7)-(9). The coefficients b, r_3 and r_5 multiplied by the absolute value of the transfer function $W(\lambda)$ at $i\omega, 3i\omega$ and $5i\omega$, respectively, are compared. If $|r_3 W(3i\omega)| \ll |bW(i\omega)|$ and $|r_5 W(5i\omega)| \ll |bW(i\omega)|$, then the subharmonics can be neglected.

The effect of the spring characteristic is investigated, so the input signal of the transfer function is the alongation and the output signal is the force in the spring:

$$W(\lambda) = -\frac{x_{in}}{x_{out}} = -\frac{1}{b}\,. \tag{10}$$

Figure 4 shows the absolute value of the product of the subharmonics and the transfer function, or more precisely, this value divided by the absolute value of the product of the harmonic and the transfer function, which is 1, as it follows from equation (10).

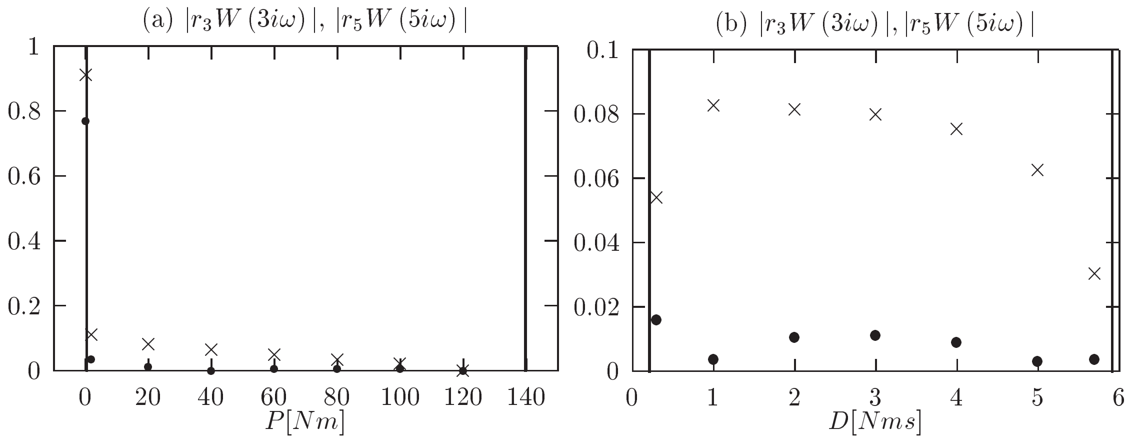

Fig.4. $|r_3W(3i\omega)|$ (crosses) and $|r_5W(5i\omega)|$ (dots)
(a) $D = 2$ [Nms], (b) $P = 20$ [Nm]

If P is chosen from a tiny region near the border of the stability domain at P_0, then subharmonics cannot be neglected, but then $|r_3W(3i\omega)|$ and $|r_5W(5i\omega)|$ decrease below 10% of $|bW(i\omega)|$. $|r_3W(3i\omega)|$ has a maximum depending on D, but even this maximum value does not reach 0.1, unless P is very close to P_0. r_5 changes its sign at a certain value of B, so $|r_5W(5i\omega)|$ has two maxima.

Concluding remarks

Results obtained by the harmonic balance method show good agreement with the earlier results presented in [12]-[13] in the major part of the stability domain. However, there are some differences for the values of P chosen near P_0. According to the results of [12], periodic solution appears at a homoclinic bifurcation point and stable fixed points are found in the small interval between P_0 and the bifurcation point, while periodic solution is found in the whole domain of stability using the harmonic balance method.

Subharmonics are neglected and it causes some problems for the values of P close to P_0. Harmonic balance method does not give reliable results here, but apart from this region, this approximation can be applied for determining the properties of periodic solutions.

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Appendix

The following parameter values are used in computations:

$m = 0.169$ [kg]	$r_w = 0.02$ [m]
$M = 1.136$ [kg]	$R_w = 0.03$ [m]
$m_m = 0.2$ [kg]	$r_m = 0.01$ [m]
$g = 9.81$ [m/s^2]	$s = 10000$ [N/m]
$l = 0.5$ [m]	$f = 0.001$ [m]
$K = 0.01$ [Nms]	

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SUMMARY

The linear stability analysis of an inverted pendulum-cart model constructed in previous works is summarized briefly. Backlash at the driving is considered by nonlinear spring characteristic. This makes the system piecewise linear and causes the oscillation of the stick around its upper equilibrium. Periodic solutions are investigated by the harmonic balance method, their amplitude and frequency are computed depending on the control parameters. Effects of higher harmonics are also estimated. Results are compared with those obtained in previous works and a region of parameters is determined where this approximation can be applied in a reliable way.