DIGITAL CONTROLLING OF PIECEWISE LINEAR SYSTEMS

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Abstract

A mechanical model of a digital balancing system is constructed and its stability analysis is presented. This model describes practical problems like backlash and sampling delay. Stationary and periodic solutions are determined numerically for the case of the system without sampling. The existence and stability of periodic solutions are checked analitically. Adding sampling delay to the system, the stability conditions change and above a critical value of the delay, the balancing is impossible. The stability chart is determined again, and stable motions are identified.

1 Introduction

Unstable equilibria of mechanical systems often have to be stabilized by control force. A number of applications can be found in this field, e.g. thrust control of aircraft, the articulated bus running on icy road, the shimmying wheel or the balancing of standing and walking robots. A typical example of stabilization of unstable equilibria is the balancing. The simplest model of balancing is that of the inverted pendulum, so it has extensive specialist literature either about its theorotical or experimental aspects [1]-[6].

A digital balancing of the inverted pendulum is examined in the subsequent chapters. The pendulum is placed on a cart, its angle and angular velocity is measured and the control force is determined by them in a way that the upper equilibrium should be stable. Control parameters must be chosen from a bounded region of the parameter plane. The control is performed by a computer and time delay occurs in the system due to the sampling of the digital processor. It causes the additional decrease of the stability domain.

The control force is provided by a motor which is also situated on the car. It drives one of the wheels of the cart through an elastic teeth belt. Backlash occuring at the driving belt makes the system piecewise linear. Stable periodic solutions appear instead of stable stationary solutions.

Many mechanical problems lead to strongly nonlinear systems due to the piecewise linear terms in their governing equations of motion. Gear pairs with backlash, impact dampers and adjacent structures during earthquakes are only few examples. The importance of this phenomenon explains the high number of publications which appear in this field [7]-[11]. An analysis is presented in [9]-[10] for determining the steady state response of piecewise linear systems and investigating its stability. A similar method is used here for the examination of periodic solutions.

2 The mechanical model and the linear stability analysis

The above mentioned digital balancing system can be seen in Figure 1 [12]-[13]. The system has 2 degrees of freedom described by the angle φ of the pendulum and the alongation $\Delta = r_m\psi - \frac{r_w}{R_w}x$ of the belt. The velocity $\dot{x}$ of the cart and the angle φ of the pendulum with its derivative are detected, but the differential gain of the cart eliminates the damping K of the motor, so the control force has the simplified form

$$Q = P\varphi + D\dot{\varphi}\,. \tag{1}$$

The linearized equations of motion assume the form

$$\begin{pmatrix} \frac{(m+M)m_m r_m}{2} & -\frac{m m_m l r_m r_w}{4R_w} \\ 0 & \frac{ml^2}{3} - \frac{m^2 l^2}{4(m+M)} \end{pmatrix} \begin{pmatrix} \ddot{\Delta} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} (m+M)\frac{K}{r_m} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\Delta} \\ \dot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{mgl}{2} \end{pmatrix} \begin{pmatrix} \Delta \\ \varphi \end{pmatrix}$$
$$+ \begin{pmatrix} (m+M)\,r_m + \frac{m_m r_m r_w^2}{2R_w^2} \\ \frac{m l r_w}{2(m+M)R_w} \end{pmatrix} R_s = \begin{pmatrix} (m+M)\,Q \\ 0 \end{pmatrix}, \tag{2}$$

where $R_s = s\Delta$ is the force in the elastic belt.

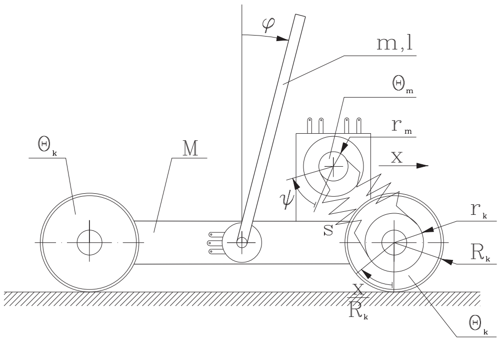

Figure 1: The inverted pendulum on a cart

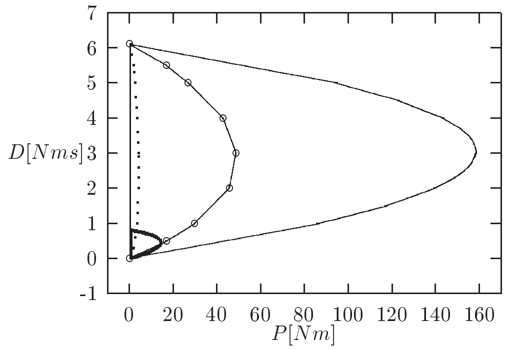

Figure 2: The stability charts with the bifurcation curves

The stability analysis is carried out by the Routh-Hurwitz criterion. The trivial solution of (2) is asymptotically stable if and only if

$$P > P_0 = \left(m + M + \frac{1}{2} m_m \frac{r_w^2}{R_w^2}\right) g \frac{r_m R_w}{r_w}$$
$$\text{and} \quad H_2 > 0\,, \tag{3}$$

where H_2 is the maximum sized Hurwitz-determinant, not presented here algebraically. The stability chart is constructed as it is shown in Figure 2.

3 The pendulum-cart model with backlash

Backlash appears in the system as a nonlinear spring characteristic. The force in the spring is the function of Δ

$$R_s = \begin{cases} s\,(\Delta + r_0) & \Delta \leq -r_0 \\ 0 & |\Delta| < r_0 \\ s\,(\Delta - r_0) & \Delta \geq r_0 \end{cases}\,, \tag{4}$$

where r_0 is the value of backlash.

New constant expressions appear in the equations of motion, this means shifting of the solutions. The stability domain does not change but it is valid only if $|\Delta| > r_0$. Otherwise, the control force is not transmitted by the belt in the tiny zone $|\Delta| < r_0$.

3.1 The bifurcation analysis

Numerical analysis of this model was accomplished in earlier works [14]-[15]. The spring characteristic has a noncontinuous first derivative and this caused problems during the numerical calculations. The bifurcation analysis was carried out by using approximate spring characteristics and the bifurcation diagrams were constructed.

After the bifurcation analysis carried out by AUTO, the stability chart in the plane of the control parameters can be constructed as it can be seen in Figure 2. It is bordered with the same straight line and parabola as it was bordered in case of the linear system (the system without backlash). Fix points are stable in a little domain near the straight line. Stable limit cycle appears at the homoclinic bifurcation point indicated with the dotted line. Fix points lose their stability at the other bifurcation point indicated with the smashed line, so all the fix points and the limit cycle are stable between the dotted and the smashed line, and only the limit cycle is stable in the remaining part of the stability domain.

3.2 The nonlinear stability analysis of periodic solutions

In case of one of the approximate spring characteristics, unstable periodic solutions appeared for values of P to the right of the smashed line in Figure 2, while the fix points were stable [15]. The amplitude of the periodic solution decreased as the approximation was more and more accurate, thus the fixed points were supposed to be unstable without limit cycles around them. The analysis presented below [10] helps to confirm the correctness of earlier results.

3.2.1 Existence of periodic response:

Let us arrange equation (2) in the following form

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\dot{\mathbf{y}} + \mathbf{S}\mathbf{y} = \tilde{\mathbf{f}}\,, \tag{5}$$

where $\mathbf{y} = \begin{pmatrix} \varphi & \Delta \end{pmatrix}^T$ and $\tilde{\mathbf{f}}$ is the vector containing the constant expressions. Then applying the Cauchy-transformation the equations of motion have this form

$$\dot{\mathbf{z}} = \tilde{\mathbf{A}}\mathbf{z} + \mathbf{f}\,, \tag{6}$$

where $\mathbf{z} = \begin{pmatrix} \varphi & \Delta & \dot{\varphi} & \dot{\Delta} \end{pmatrix}^T$.

The general solution of equation (5) can be written in the following form

$$\mathbf{z} = \boldsymbol{\phi}\boldsymbol{\gamma} + \boldsymbol{\xi}\,, \tag{7}$$

where $\boldsymbol{\gamma}$ contains the constants which can be determined from the initial conditions.

One type of periodic solution intersects both borders of backlash. This periodic solution is symmetric and must satisfy the periodicity and the matching conditions

$$\mathbf{z}_1(0) = -\mathbf{z}_2(t_2)\,, \quad \mathbf{z}_1(t_1) = \mathbf{z}_2(0) \tag{8}$$

$$\Delta_1(0) = \Delta_2(0) = r_0\,. \tag{9}$$

Index 1 concerns the zone $|\Delta| > r_0$, while index 2 concerns the zone $|\Delta| < r_0$. t_1 and t_2 are the time spent in the corresponding zone in half a period. $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, t_1$ and t_2 are 10 unknowns, while equations (8) and (9) represent 10 equations. However, t_1 and t_2 cannot be expressed explicitly, so the unknowns can be determined on the basis of the following process.

The amplitude of periodic solutions obtained by using the approximate spring characteristics depends on the correctness of the approximation, but their period is the same if all the other parameters are fixed. Thus t_1 and t_2 are determined numerically and used in equation (8) to compute $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$. Then the obtained result is checked by equation (9).

The other type of periodic solution intersects only one of the borders of backlash. The same process can be applied, but the periodicity conditions are changed

$$\mathbf{z}_1(0) = \mathbf{z}_2(t_2)\,, \quad \mathbf{z}_1(t_1) = \mathbf{z}_2(0)\,. \tag{10}$$

3.2.2 Stability of periodic response: Let a perturbed periodic solution be $\bar{\mathbf{z}}_1$, the deviations of $\bar{\mathbf{z}}_1$ and $\bar{\mathbf{z}}_2$ from $\mathbf{z}_1$ and $\mathbf{z}_2$ at time t_1 and t_2 are $\delta\mathbf{z}_1$ and $\delta\mathbf{z}_2$, and the deviation from the correct initial conditions is $\delta\mathbf{z}_0$. The relation between the perturbations at the beginning and at the end of the first period can be written in the following form [10]

$$\delta\mathbf{z}_2 = \mathbf{\Pi}\delta\mathbf{z}_0\,. \tag{11}$$

The periodic solution is stable if and only if the absolute value of all the eigenvalues of $\mathbf{\Pi}$ is less than 1.

The earlier results are confirmed by applying this analysis. Stable periodic solutions are found in the majority of the stability domain as it is shown in Figure 2. The amplitude of unstable periodic solutions found by the numerical analysis using approximate spring characteristics is zero, so the assumption that fix points are unstable and unstable periodic solutions do not exist is right.

4 The digital effect

The digital effect causes time lag in the control force. To emphasize it, the equation of motion (6) is written in the form

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{b}u(t) + \mathbf{f}\,, \quad u(t) = \mathbf{D}\mathbf{z}(t-\tau)\,, \tag{12}$$

where $\mathbf{D} = \begin{pmatrix} P & 0 & D & 0 \end{pmatrix}$ and τ is the time delay. Let us introduce the following nominations

$$\mathbf{A_d} = e^{\mathbf{A}\tau}\,,$$

$$\mathbf{b_d} = \left(\int_0^\tau e^{\mathbf{A}t}dt\right)\mathbf{b}\,, \quad \mathbf{f_d} = \left(\int_0^\tau e^{\mathbf{A}t}dt\right)\mathbf{f}\,, \tag{13}$$

and then the discrete equations of motion have this form

$$\mathbf{z}_{n+1} = \mathbf{A_d}\mathbf{z}_n + \mathbf{b_d}u_n\,, \quad u_{n+1} = \mathbf{D}\mathbf{z}_n\,. \tag{14}$$

Merging $\mathbf{z}$ and u in one vector

$$\boldsymbol{\zeta} = \begin{pmatrix} \mathbf{z} \\ u \end{pmatrix}, \quad \mathbf{\Phi} = \begin{pmatrix} \mathbf{A_d} & \mathbf{b_d} \\ \mathbf{D} & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \mathbf{f_d} \\ 0 \end{pmatrix}, \tag{15}$$

the discrete equations of motion can be obtained in the following form

$$\boldsymbol{\zeta}_{n+1} = \mathbf{\Phi}\boldsymbol{\zeta}_n + \boldsymbol{\beta}\,. \tag{16}$$

Neglecting backlash ($\boldsymbol{\beta} = \mathbf{0}$) the upper equilibrium of the pendulum is stable if and only if the absolute value of all the eigenvalues of $\mathbf{\Phi}$ is less than 1. After applying the Moebius-transformation, the Routh-Hurwitz criterion can be used for determining the stability conditions. The stability chart is given in Figure 2 for $\tau = 0.005[s]$ with the thick line.

Adding backlash to the system, the perfect stabilization of the upper equilibrium is impossible, but 2 kinds of stable motion are obtained. One is drawn in Figure 3(a) which can be found only in a very little domain for the smallest values of the control parameters in the stability domain. $P = 0.2[Nm]$, $D = 0.05[Nms]$ in the figure. Trajectories go to a stable set and move inside it, but there is not any limit cycle inside the set, so the motion looks chaotic. Since the phase-space is symmetric, the same kind of motion exists around the other equilibrium. Another type of chaotic motion occurs for greater values of the control parameters. $P = 5[Nm]$, $D = 0.5[Nms]$ in Figure 3(b) where this type of motion is shown. Note that the presented phase-diagrams are results of simulations, thus chaos may be caused by numerical problems. Further efforts are needed to identify the reason of chaos.

The stability domain shrinks as the sampling delay increases and above a critical value, the balancing is

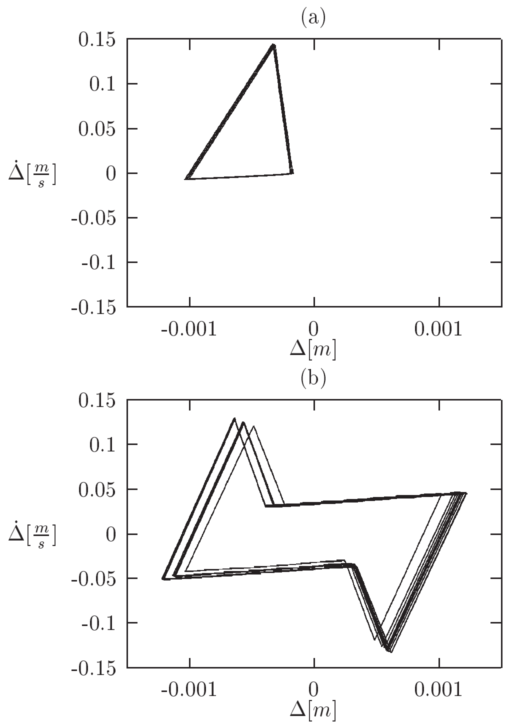

Figure 3: The 2 kinds of stable motion

impossible. The critical time delay depends on the parameters describing the system [14].

5 Conclusions

Increasing time delay tends to destabilize the digitally controlled dynamical systems. Backlash causes oscillations of the pendulum around its upper equilibrium. Numerical and analytical results show good coincidence about its effect. Occurance sampling delay and backlash together may lead to chaotic motion. The examined model is an example for the problems of digital stabilization of unstable equilibria of piecewise linear systems, but the main principles and methods are valid for the stabilization of unstable equilibria of any other controlled piecewise linear systems.

Acknowledgments
This research was supported by the Hungarian Scientific Research Foundation under grant no. OTKA T030762 and the Ministry of Culture and Education under grant no. FKFP 0380/97.

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