Structure of Mathematical Morphology

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ABSTRACT

For contact surface measurement, estimating the profile of actual surface from the locus of the center point of the stylus is an inverse problem. The measurement process which transforms the profile of actual surface to the locus of the center point is a dilation operation with a disk with error. An erosion operation can be taken as the inverse mapping of the dilation to estimate the actual profile. By using category theory, erosion and dilation can be formulated as two functors between two categories of sets of vectors. Each category has sets of vectors as objects and inclusion functions between sets as morphisms. The functor of erosion is left adjoint to the functor of dilation.

Keywords Mathematical morphology, surface metrology, inverse problem, adjoint functors

1 An Inverse Problem in Surface Metrology

For contact surface measurement, a small stylus is used to measure the surface profile. The measurement process can be considered as a disc S of radius r rolling over the measured profile, as shown in figure 1. That is a mapping G: $L \rightarrow C$ transforms the actual profile, l say, to the locus of the centre point of S, curve c say, where L is the set of the surface profiles, C is the set of the loci of the center points. Estimating the actual profile from the measurement result c is an *inverse problem*, which can be solved by finding the proper inverse mapping of G, written as $G^{\dagger}$: $C \rightarrow L$. Obviously, the shape of curve c is different from the actual profile (refer to figure 2), thus the forward mapping G is not simply a linear function that increases the height of l by r. In section 2, we'll show that G and $G^{\dagger}$ can be modeled as morphological operations.

It can be observed from figure 2 that small changes of l at certain section of the gap or the inner corner will not change the loci of the center points. Thus, for certain measurement results c, there could be more than one surface profile that can fit c exactly, that means the inverse solution is not unique.

2 Dilation and Erosion

Erosion and dilation are the basic operators of mathematical morphology. The erosion of A by a structuring element S, such as a disk, can be understood as the loci of points reached by the centre of S when S moves inside A; and the dilation of A by S can be understood as the loci of points covered by S when the centre of S moves inside A (See figure 3).

In Euclidian space E^3, assume the projection of the measured profile in x-y plane is a straight line. Let the intersection surface of the measured part, intersected along the measured profile l by x-z plane, be a set A in E^2. Let R be a fixed rectangular in x-z plane, such that c and l are included in R and the projections of R and l in x-y plane are the same. Let $A_R = A \cap R$ (see figure 1). The spherical measuring stylus can be taken as a structuring element S which is a disk in x-z plane. Let Ds: $M \rightarrow D$ be a dilation operator and Es: $D \rightarrow M$ be the erosion operator with structuring element S, where M and D are two partial ordered sets (poset) with elements as sets in E^2 ordered by set inclusion.

By the duality of the erosion operator and the dilation operator, $[Ds(A)]^{C} = Es(A^C)$, thus $Ds(A) = [Es(A^C)]^C$, where A^C is the complement of A. If there is no noise in c, it's easy to imagine that the boundary of $Es(A^C)$ in R is coincident with the measurement result c.

Let π be a function that extracts the upper boundary line of a region (set in E^2), as shown in figure 4. Then we have, $\pi(A_R) = l$, and $\pi(D_S(A) \bigcap R) = c$. Let π^* be a function that transforms a curve (or line) l in R to a region (set of points) below l and in R (see figure 4), such that $\pi^* \circ (\pi \circ \bigcap_R) = I$, that means $\pi^*(l) = A_R$.
The relation between input L and output C can be described by the below commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "Ds"] & D \arrow[r, "Es"] \arrow[d, "\cap_R"] & M \arrow[d, "\cap_R"] \\
 & D_R \arrow[d, "\pi"] & M_R \arrow[d, "\pi"] \\
L \arrow[uu, "\pi^*"] \arrow[r, "G"] & C \arrow[r, "G^\dagger"] \arrow[uu, "\pi^*", shift left=4ex] & L
\end{tikzcd}$$

That means, $G = \pi \circ \bigcap_R \circ D_S \circ \pi^*$ and $G^\dagger = \pi \circ \bigcap_R \circ E_S \circ \pi^*$. (1) & (2)
Therefore, by defining a proper coordinate and interested region R, the forward mapping G can be formulated as a customized dilation, and the inverse mapping $G^\dagger$ can be formulated as a customized erosion.

3 Adjunction

Take the two posets M and D as two categories of poset, written as **M** and **D**. The objects in **M** and **D** are sets in E^2 ordered by inclusion. For any objects A, B in **M** or **D**, there is a unique morphism from A to B if and only if $A \subseteq B$. The sets in **M** are determined by the (actual or estimated) surface profiles, and the sets in **D** are determined by the loci of center point as shown in section 2.
According to the properties of erosion and dilation, we have:

(1) Erosion and dilation are increasing, i.e. order preserving.
Hence they can be considered as two functors between **M** and **D**.
(2) $A \subseteq E_S D_S(A) = A_C$, for all $A \in \mathbf{D}$. A_C is the closing of A by S.
(3) $B \supseteq D_S E_S(B) = B_O$, for all $B \in \mathbf{M}$. B_o is the opening of B by S.
(4) $A \subseteq E_S(B) \Leftrightarrow D_S(A) \subseteq B$, i.e. $\hom(A, E_S(B)) \cong \hom(D_S(A), B)$.
That is a bijective relation of mappings of two categories.

Thus we have the structure shown in the diagram of figure 5, which is exactly an adjunction. That means (D_S, E_S) is an adjoint pair, i.e. the inverse mapping is left adjoint to the forward mapping. By the property of adjunction, we have:

(5) $D_S E_S D_S = D_S$ and $E_S D_S E_S = E_S$.

Thus, by equation (1) & (2),

$$\begin{aligned}
GG^\dagger G &= \pi \circ \bigcap_R \circ D_S \circ \pi^* \circ \pi \circ \bigcap_R \circ E_S \circ \pi^* \circ \pi \circ \bigcap_R \circ D_S \circ \pi^* \\
&= \pi \circ \bigcap_R \circ D_S \circ E_S \circ D_S \circ \pi^* = \pi \circ \bigcap_R \circ D_S \circ \pi^* = G, \\
G^\dagger G G^\dagger &= \pi \circ \bigcap_R \circ E_S \circ \pi^* \circ \pi \circ \bigcap_R \circ D_S \circ \pi^* \circ \pi \circ \bigcap_R \circ E_S \circ \pi^* \\
&= \pi \circ \bigcap_R \circ E_S \circ D_S \circ E_S \circ \pi^* = \pi \circ \bigcap_R \circ E_S \circ \pi^* = G^\dagger,
\end{aligned}$$

i.e. $GG^\dagger G = G$ and $G^\dagger G G^\dagger = G^\dagger$. That means $G^\dagger$ is the non-linear generalized inverse of G.
As mentioned in section 1, there can be many surface profiles fit a certain measurement result c. That is because dilation is not a injective mapping. By property (2), A_c includes all the sets

fit a certain set bounded by locus c. Thus the inverse solution l_c estimated by $G^\dagger$ is the highest surface profile which fits the locus c.

4 Conclusions

The inverse problem of contact surface measurement is formulated by the customized erosion and dilation. Dilation and erosion is an adjoint pair. From the adjoint structure of erosion and dilation, the relation of forward and inverse mappings of this inverse problem is derived. Moreover, if the measurement result c contains not noise, since $GG^\dagger G = G$, c must satisfies the equation $GG^\dagger(c) = c$. Thus $GG^\dagger$ of this inverse problem can be used to identify and filter out the noise of measurement result. That's the principle of applying the combination of dilation and erosion (opening and closing) as morphological filters, which appeared in ISO 16610/DTS.

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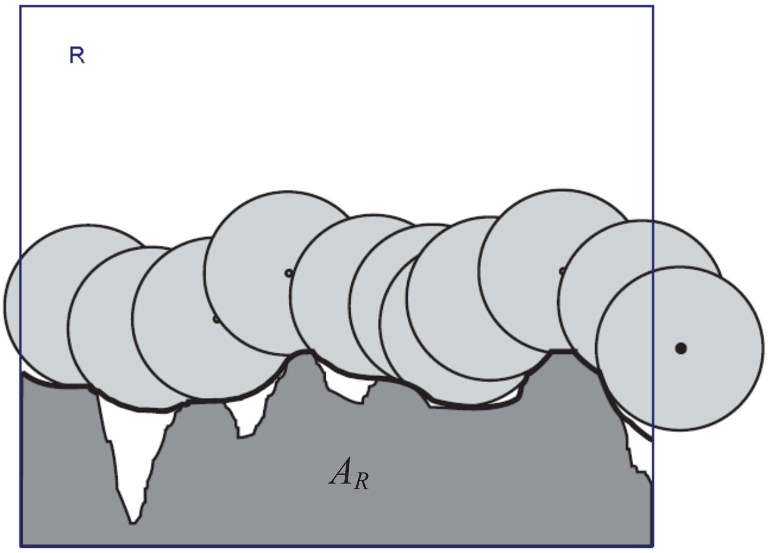


Figure 1: Contact surface measurement

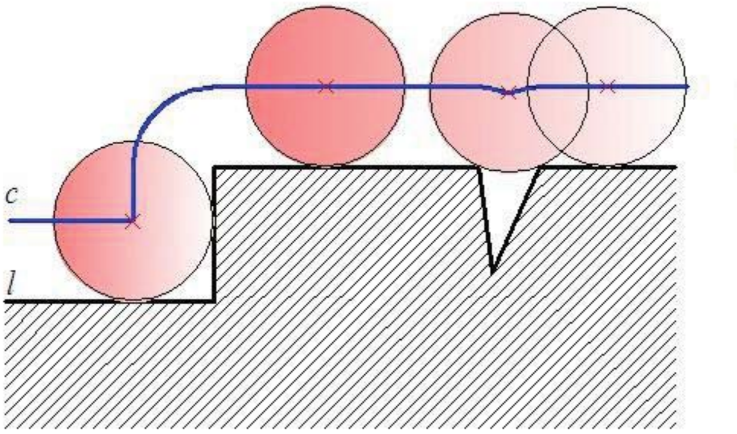
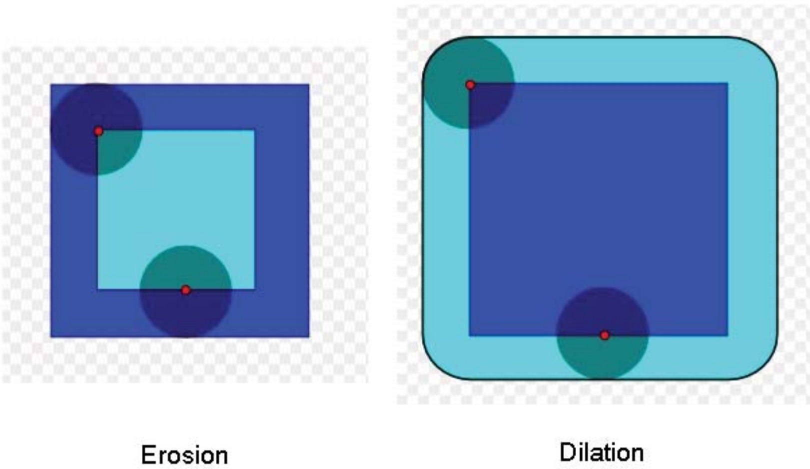


Figure 2: Actual profile and locus of center point



The erosion of the dark-blue square by a disk, result in the light-blue square.
The dilation of the dark-blue square by a disk, result in the light-blue square with rounded corners.

Figure 3: Erosion and Dilation

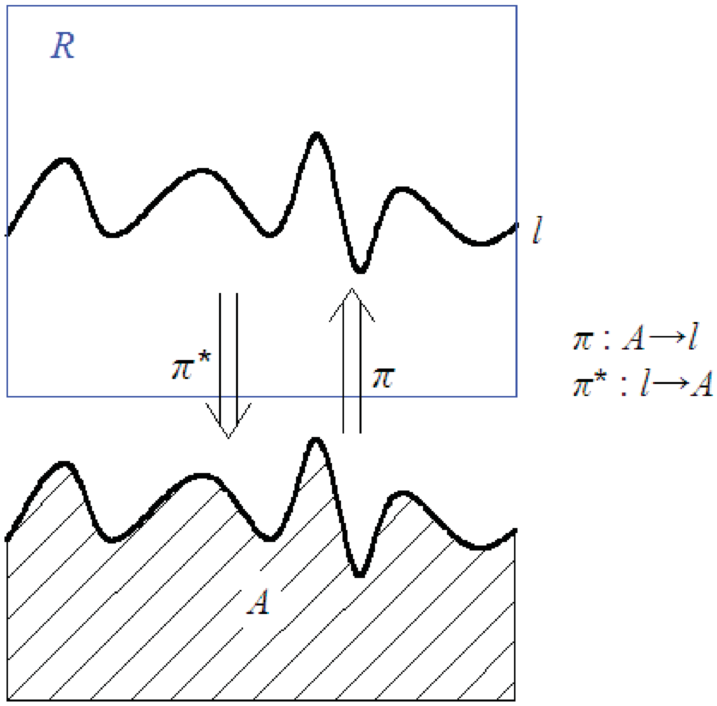


Figure 4: functions between curve and region

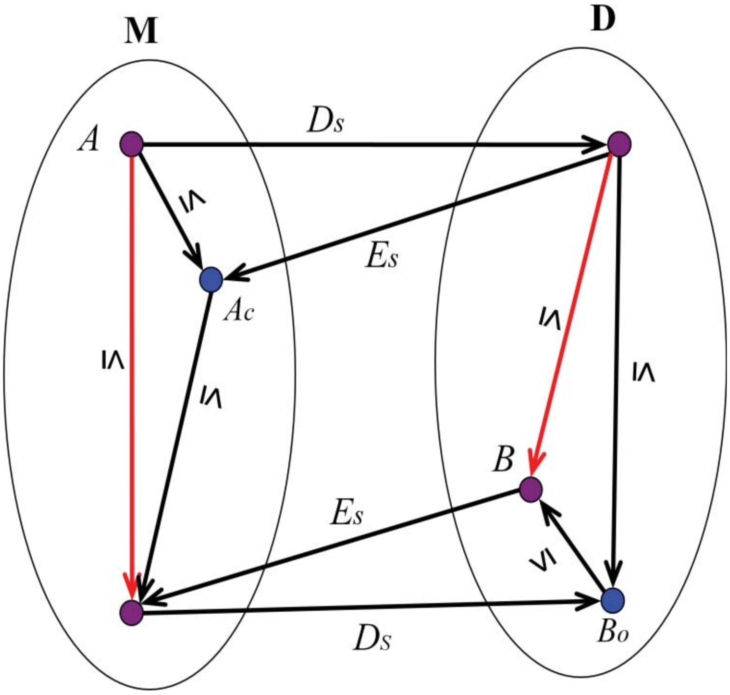


Figure 5: Adjoint structure of Erosion and Dilation