An experimental method to measure gear tooth stiffness throughout and beyond the path of contact

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Abstract: A method is presented that allows the accurate measurement of the tooth pair stiffness of a pair of spur gears. The method reveals the stiffness behaviour throughout the full length of the normal path of contact and also into the extended contact region when tooth corner contact occurs. The method makes use of the properties of transmission error plots for mean and alternating components over a range of tooth loads (Harris maps). It avoids the usual problem when measuring tooth deflections that deflections of other test rig components are difficult to eliminate. Also included are predicted Harris maps for a pair of high contact ratio spur gears, showing the effects of various simplifying assumptions, together with a measured map.

Keywords: gear tooth stiffness, gear tooth relief, gear transmission error, gear path of contact

NOTATION

c_b	bearing and shaft stiffness
c_t	combined single-pair tooth stiffness
N	tooth loading per unit face width
δ	measured transmission error deflection

1 INTRODUCTION

It has long been known that gear teeth deflect significantly under the considerable loads they transmit, and this gave rise to the practice of 'tip easing' to minimize dynamic loads, noise and scuffing arising from tooth corner contact. This practice was put on a scientific basis by Walker [**1**, **2**], who estimated the tooth deflection at a chosen design load and then introduced a calculated tip (and/or root) profile relief to compensate for this deflection. Harris [**3**] published a remarkable paper concerning dynamic loads in spur gear teeth, in which several important new concepts were introduced. He showed that in high-speed gearing the dynamic behaviour was a steady state vibration in which tooth deflection had two important roles:

1. It provided an important stiffness parameter in a mass–spring geared system.
2. It was a major source of excitation for the vibration system.

To describe the excitation, he introduced the concept of 'transmission error', which is a function of tooth profile and deflection under load, again pointing to the importance of tooth stiffness. The transmission error is very conveniently described by what has become known as a 'Harris map' (see reference [**4**] for a modern description and Figs 1 to 4 of this paper for examples). The several predictions of Harris were confirmed experimentally by Gregory *et al.* [**5**].

Soon after the work of Harris, Niemann and Winter [**6**] introduced two specific types of profile relief, known as 'long' and 'short', which have quite different excitation characteristics. This was then generalized by Munro and Yildirim [**7**–**10**] to show that long and short reliefs in spur gears gave the limits of the useful range, and that 'intermediate' reliefs were also useful. In particular, the type of relief was shown to determine the dynamic performance in terms of vibration and noise, confirmed experimentally by Munro and Palmer [**11**].

2 TOOTH STIFFNESS STUDIES

Some of the earliest studies were done by Weber [**12**], using strain energy methods and breaking down the deflection into a number of components, such as cantilever bending, shear, base rotation and Hertzian. Naturally, the deflection of each tooth was a maximum with the load at the tip and a minimum when near the

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root, so that the combined deflection of the tooth pair was a minimum near the pitch point. To this combined deflection must be added the Hertzian deflection, which is slightly non-linear, but since it is much smaller than the other (linear) components the non-linearity is usually ignored. The Weber work resulted in the adoption of a convenient rounded number for combined tooth stiffness in imperial units of 2000 lbf/in of face width to give a deflection of 0.001 in. The corresponding SI unit is 13.8 N/mm per micrometre. This is the stiffness at the pitch point, and it falls by about 30 per cent at the ends of the path of contact. It should be noted that the tooth stiffness is independent of module. The cross-coupling effect between teeth can usually be ignored.

Many other studies of tooth deflection have been carried out subsequently, using analytical and more recently finite element methods. A good summary of this work has been presented by Steward [**13**], together with his own finite element analysis and experimental work. He pointed out that a major difficulty in comparing different research results arose from the uncertainty of a datum from which the deflection should be measured. This gave rise to variations of more than 100 per cent in published results, and even greater if gear body deflection was included, as Steward did.

The question of a datum becomes important in considering the use of a stiffness value. For dynamic analysis it involves the specific shape of the gear body and the method of fixing the body to a shaft. Clearly this can be different in every design. For profile relief calculations the problem is much simpler, since the datum is the adjacent, non-loaded tooth, and this is the case to be considered in this paper.

Experimental validation of tooth stiffness prediction is very difficult, because the large forces needed to deflect the teeth by a significant amount will also deflect everything else in the test rig and the problem again becomes one of the datum. In the experimental study described in this paper the problem of non-tooth deflections is overcome by measuring deflections both for one-pair and two-pair tooth contact. The other deflections, such as shafts and bearings, are eliminated because they are common to the two sets of measurements for a given tooth load. The tests were assisted by making use of the high precision of measurement afforded by modern transmission error equipment. The tests were also extended to measure tooth pair stiffness beyond the normal path of contact. This has been ignored by previous researchers because of both analytical and experimental difficulties, but it is important when profile relief is insufficient to avoid tooth corner contact [**14**].

3 TEST RIG, TEST GEARS AND INSTRUMENTATION

All the experimental results were obtained on a gear test rig at the University that had previously been used for transmission error and noise analysis [**11**]. It is a standard back-to-back arrangement with a centre distance of 203.2 mm (8 in), the drive and test gearboxes being separated by a split vernier coupling allowing fixed increments of torque to be locked in the system. The tests were carried out at a speed of about 60 r/min, so that dynamic effects were negligible. The gear shafts rotate in precision taper roller bearings, which were preloaded to avoid non-linearity effects.

The transmission error is measured using a pair of Heidenhain 36 000 line encoders, each with an EXE 702B interpolation unit, and the outputs are processed using a GFM GP36 computer system. This set-up has an accuracy of better than 0.5 arcsec.

Test gear details for both the pinion and wheel are as follows:

Module	3.738
Number of teeth	54
Pressure angle	18°
Face width	11 mm
Outside diameter	212.344 mm
Root diameter	190.575 mm
Pitch circle diameter	201.854 mm
Arc tooth thickness (at pitch circle diameter)	6.220 mm
Contact ratio	2.16
Quality	BS 436 grade 2-3

The gears have zero profile relief and 3–4 μm lead crowning.

4 TRANSMISSION ERROR OF TEST GEARS

The transmission error (TE) of the gears under load can be predicted using the thin slice theory [**15–17**]. This is incorporated in a computer program developed at the University and allows the tooth pair stiffness to be varied along the path of contact. Figure 1 displays results when the stiffness is constant along the path of contact. The regions where there are three pairs of teeth in contact are the raised rectangular parts in each curve. The other regions have two pairs of teeth in contact so the stiffness is 2 and 3 times the single-tooth pair stiffness in each region respectively.

If the stiffness is varied along the path of contact to approximately two-thirds of the value at the end of the path of contact compared with that at the centre of the path of contact, the results appear as in Fig. 2. When extended contact beyond the normal path of contact is included, the results appear as in Fig. 3. Figures 1 to 3 can be compared with the actual measured values (Fig. 4), where the effects of both variable stiffness and extended contact can clearly be seen. The slight downward slope from left to right in Fig. 4 is caused by a small amount of eccentricity in the test gears. A small amount of profile error is also evident.

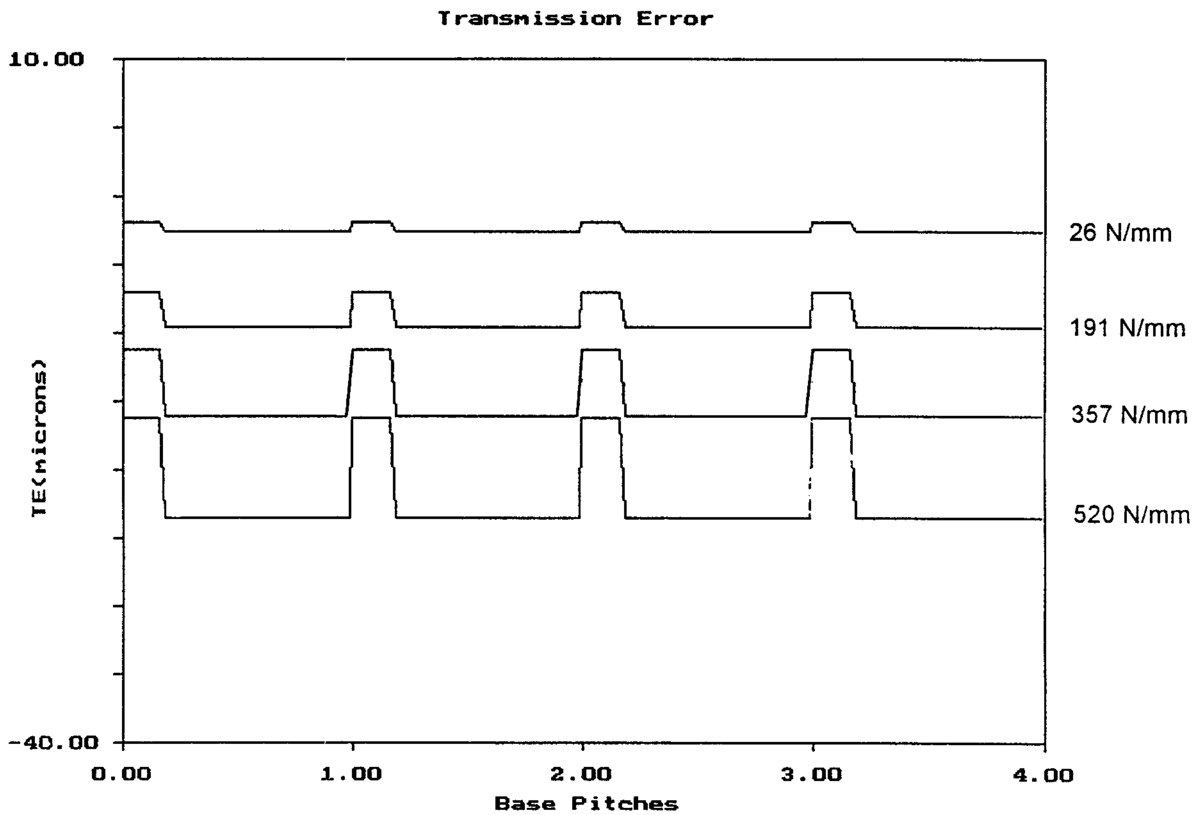

Fig. 1 Transmission error with constant stiffness (12 GPa)

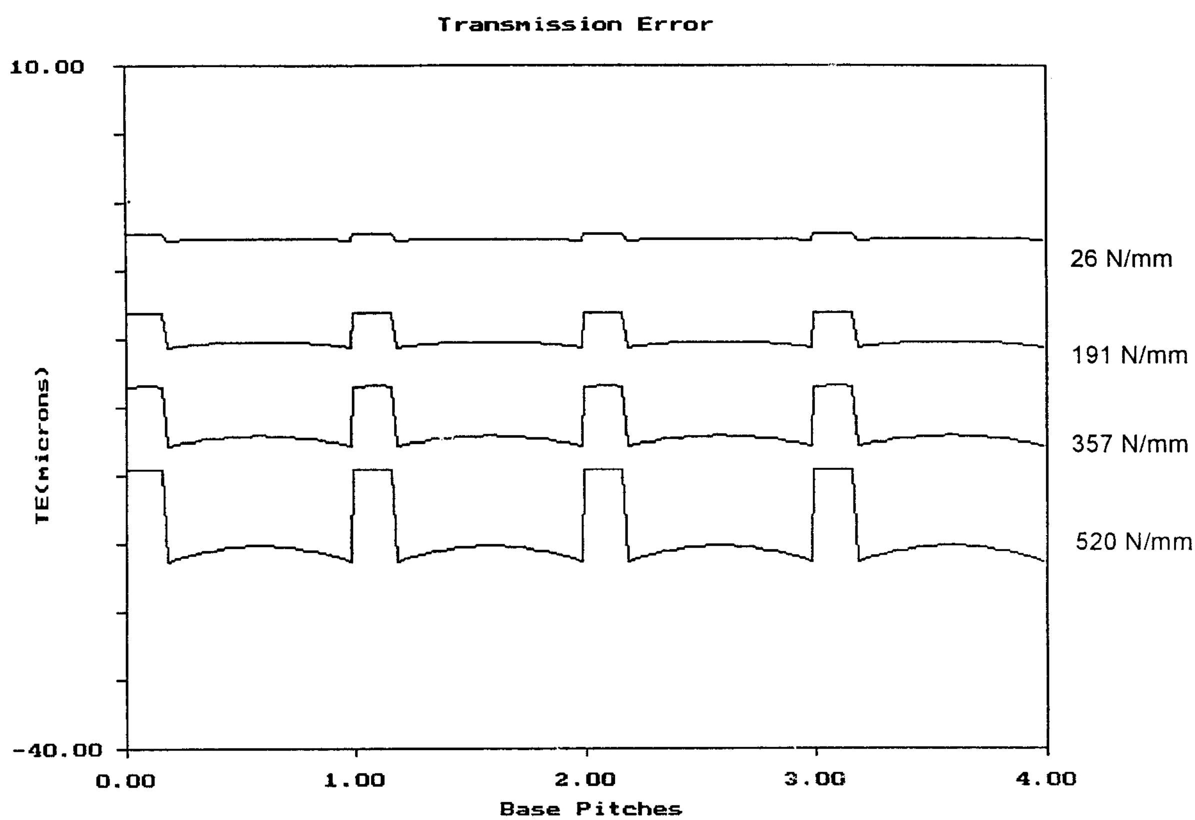

Fig. 2 Transmission error with variable stiffness (12–8 GPa)

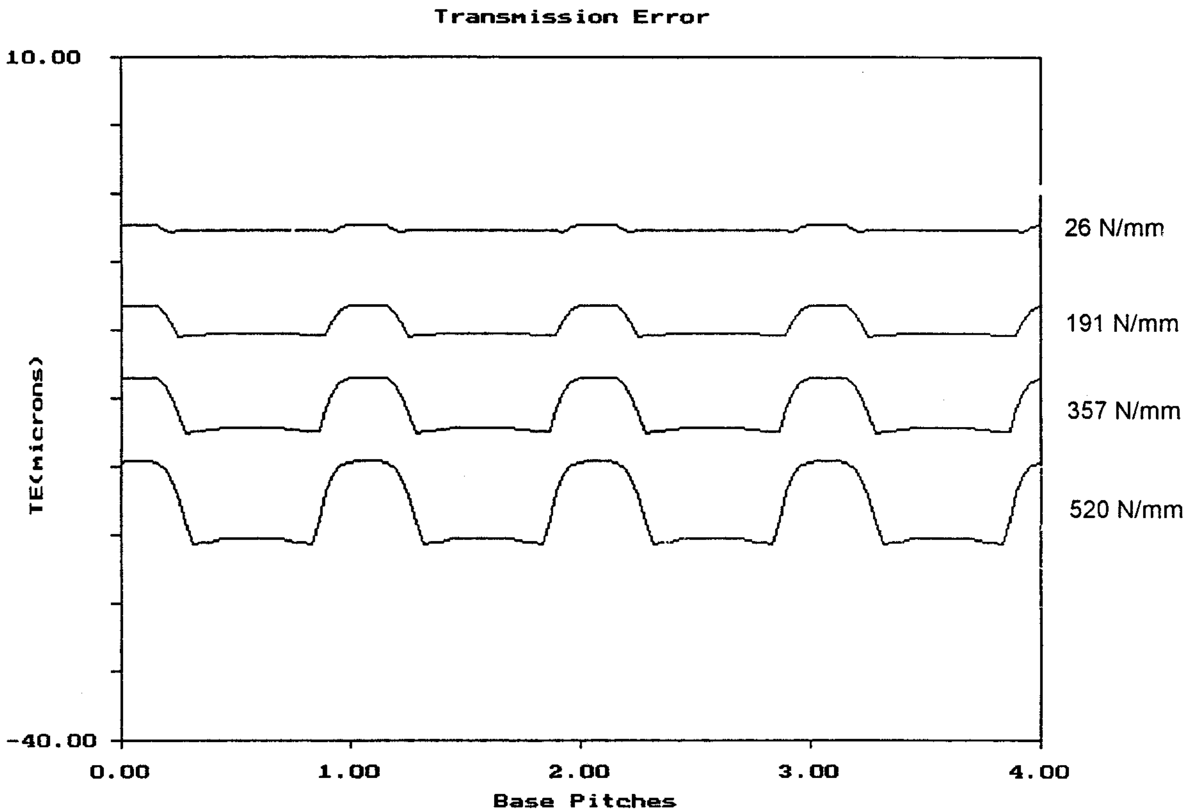

Fig. 3 Transmission error with variable stiffness and corner contact

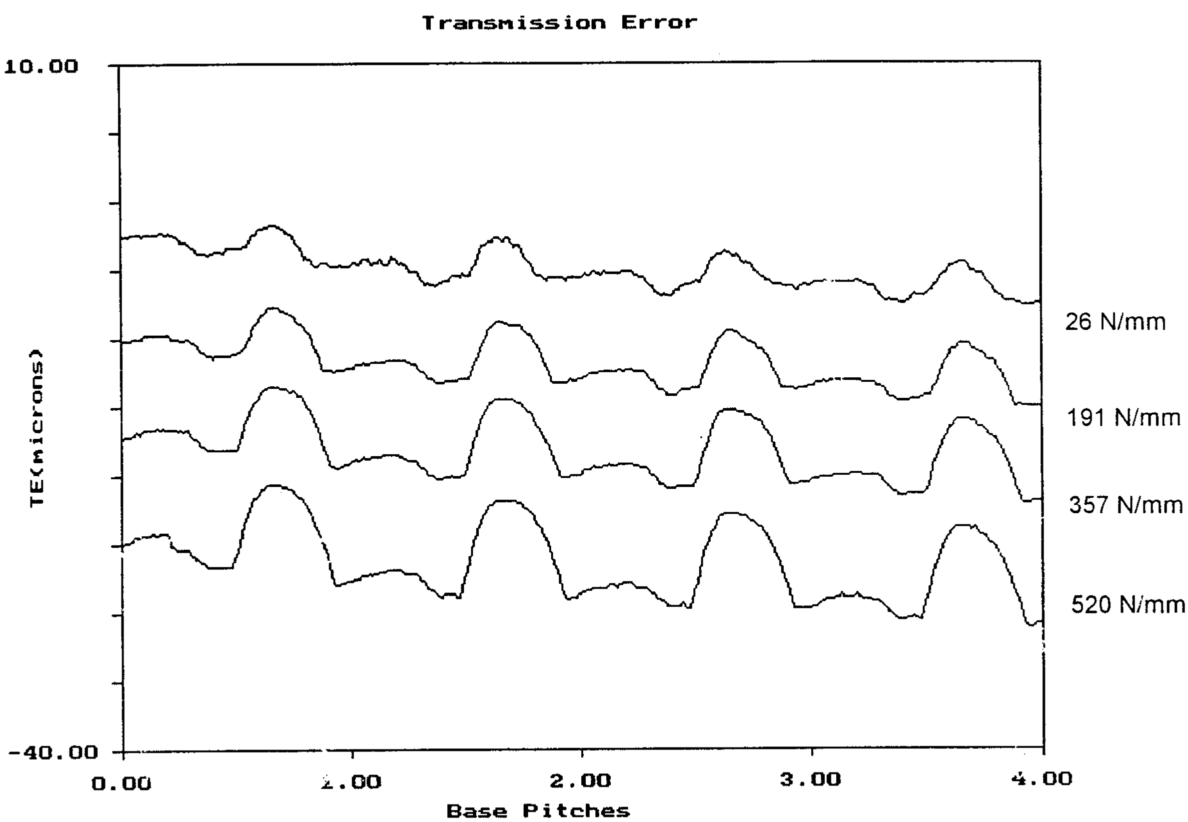

Fig. 4 Transmission error with variable stiffness and corner contact (measured data)

From the results it is obvious that the value and variation of tooth pair stiffness throughout the whole path of contact cannot be evaluated simply from the TE of the gears under normal meshing conditions. The following method, however, allows the variation in stiffness to be evaluated throughout the whole path of contact, including the tooth corner contact.

5 DESCRIPTION OF THE METHOD TO EVALUATE STIFFNESS

One method to evaluate the variation in stiffness would be to produce gears with a contact ratio of exactly 1. The problem with this, however, is that the contact outside the normal path of contact causes the effective contact ratio to be increased if no relief is applied. The other method would be to extend the centre distance. This would produce the results, but the gears would not be meshing at the same relative roll distances and angles and only the top part of the teeth would be in contact, thus giving much lower values of stiffness. Also, the length of the path of contact would be greatly reduced. A third method might be to attach a shim to one tooth face, but it would not be possible to simulate tooth corner contact.

The method used in this investigation was to remove the neighbouring teeth of a gear so that they no longer contribute to the mesh cycle and the engagement of a tooth pair can be examined throughout its whole range. Removing whole teeth might, however, change the stress pattern close to the neighbouring tooth, so reduction of the tooth width on one flank effectively to produce a negative adjacent pitch error is better. This was achieved by grinding material from the tooth surface to a depth of about 0.5 mm (Fig. 5). The contact ratio is 2.16, and therefore, in order to produce the required effect, two teeth either side of an unmodified tooth must be ground.

Figure 6 is a sketch that displays the relative positions of the paths of contact of the gear tooth pairs and the overlapping and combined tooth pair stiffnesses. The resulting mesh stiffness with teeth removed is then displayed. When the gears are meshed in practice, the phenomenon of tooth corner contact should produce contact beyond the normal path of contact. Two TE curves that result from a zero (or small finite load) and higher load are finally shown in Fig. 6. Subtracting the lower curve from the upper curve to give a difference δ, and then dividing the result into the load, should produce the stiffness value. In practice the values of δ will include the bearing and other transverse deflections such as shafts, but these deflections can be removed using the following method.

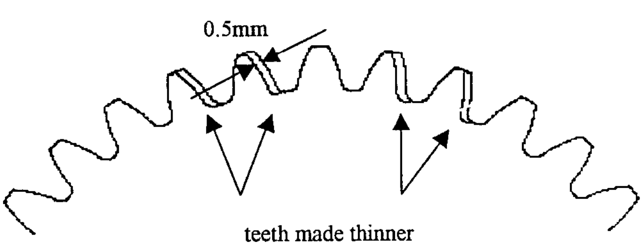

Fig. 5 Modification of a gear to allow the analysis of stiffness variation

The method makes use of the fact that in this case the two gears are identical, so that the stiffness curves are symmetrical about the pitch points. Thus, at a position mid-way between the pitch points, the stiffness values for the two pairs are the same.

The following two equations hold:

For the region with one pair in contact:

$$\delta_1 = \frac{N}{c_b} + \frac{N}{c_t} \tag{1}$$

For the region with two pairs in contact:

$$\delta_2 = \frac{N}{c_b} + \frac{N}{2c_t} \tag{2}$$

where

N = difference in applied load per unit face width
c_b = transverse stiffness due to bearings, etc.
c_t = tooth stiffness at δ_1 and δ_2 (note that δ_1 and δ_2 are at the same positions relative to their pitch points, Fig. 6)

Solving equations (1) and (2) for bearing stiffness and the tooth pair stiffness at the positions of δ_1 and δ_2 gives

$$c_b = \frac{N}{2\delta_2 - \delta_1} \tag{3}$$

$$c_t = \frac{N}{2(\delta_1 - \delta_2)} \tag{4}$$

The bearing stiffness is sensibly constant along the path of contact in relation to the tooth stiffness, so that its effect on deflection δ_1 can be allowed for, to give the single pair tooth stiffness for any position of δ_1 along the path of contact.

The method could also be applied to non-1:1 ratio gears, but the process becomes slightly more complex. The overlapping stiffnesses at the position of δ_2 can no longer be assumed to be equal, so a third equation is required to calculate the stiffnesses. This is obtained from the position of δ'_1, which is one pitch away.

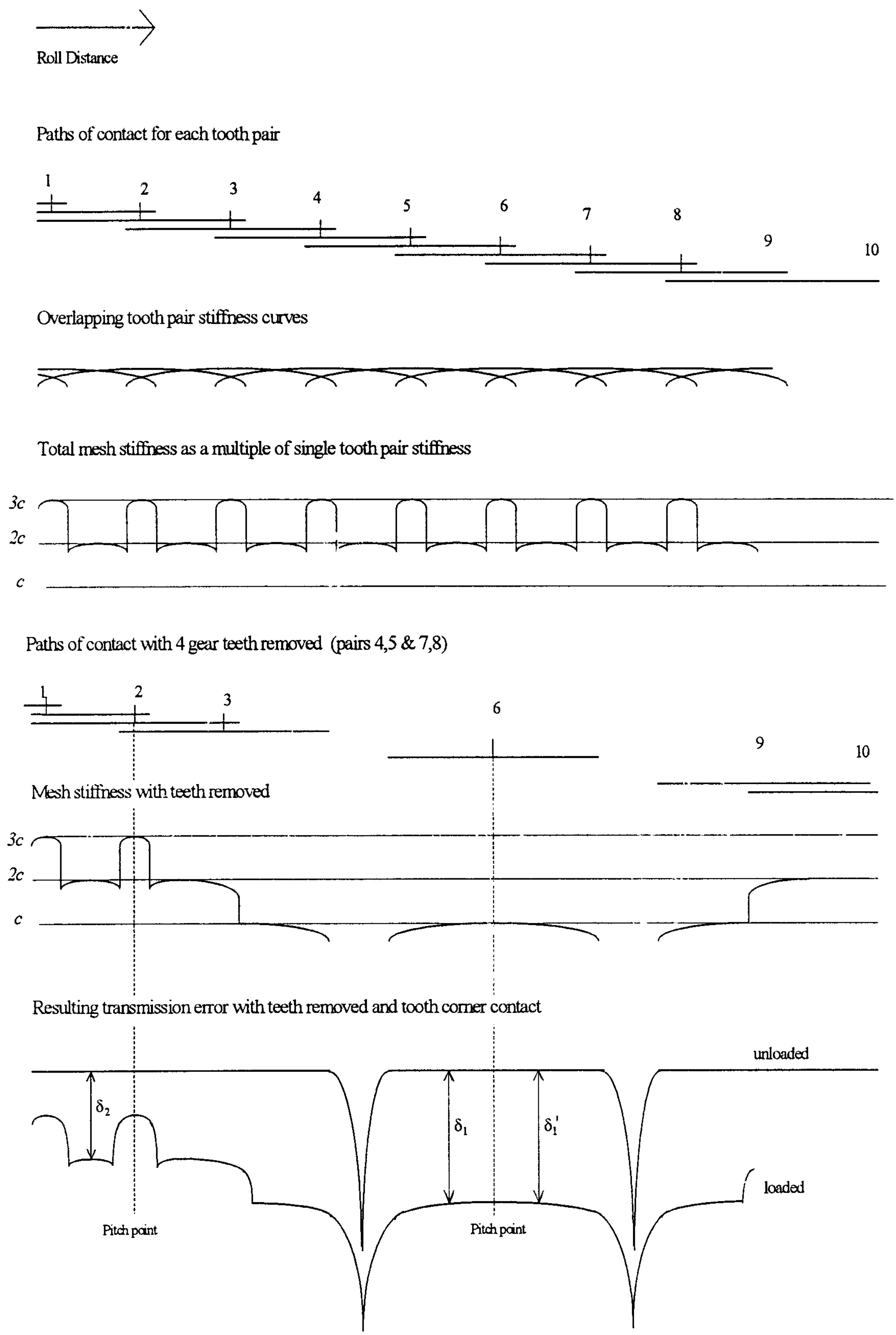

Fig. 6 Theoretical stiffness and transmission error of a pair of high contact ratio gears

The three equations to be solved would then be

$$\delta_1 = \frac{N}{c_b} + \frac{N}{c_t} \tag{5}$$

$$\delta'_1 = \frac{N}{c_b} + \frac{N}{c'_t} \tag{6}$$

$$\delta_2 = \frac{N}{c_b} + \frac{N}{c_t + c'_t} \tag{7}$$

where c'_t is the tooth pair stiffness at the position of δ'_1.

6 EXPERIMENTAL RESULTS

The TE curves presented in Fig. 7 are the theoretical predictions made on the model developed at Huddersfield [**17**]. The top curve is the TE for a light load, 26 N/mm, and the curve below this is the curve for a load of 520 N/mm. The parabolas for contact outside the normal path of contact can be seen and also slight differences in approach and recess, the former being slightly steeper, which is consistent with developments in reference [**14**].

Experiments with the 1:1 ratio test gears described in Section 3, with teeth removed as explained in Section 5, were conducted, and the measured results for the same two loads (26 and 520 N/mm) are shown in Fig. 8. They display a close similarity to the predicted values in Fig. 7 and also prove that there is a difference in extended contact for approach and recess even in 1:1 ratio gears [**14**].

The top curve of Fig. 9 shows the difference in TE for the two loaded curves for one tooth pair only. This difference is used to calculate the tooth pair stiffness. The dashed line represents the region of extended contact.

From Fig. 9, the values of δ_1 and δ_2 were taken and, using equation (4), the bearing and other element stiffness c_b was found to be 15.85 GPa. (The unit GPa is used throughout this paper, although it is usually more useful to think of tooth loading in terms of N/mm of face width to give a tooth deflection of 1 μm. Numerically they are the same.) When this value is taken into account, the 'pure' tooth pair stiffness can be found and is shown in Fig. 10. It varies in a parabolic manner as expected [**12, 13**], although the values at the pitch point are slightly higher than previously thought for this gear pair.

The stiffness outside the extended path of contact starts as an extrapolation and then seems to level off. This may be explained by the direction of the line of action of the force changing to act down the tooth length as the tooth rolls round on the tip rather than acting somewhat across the tooth to bend it. The sharp increases in the stiffness values at each end of the trace

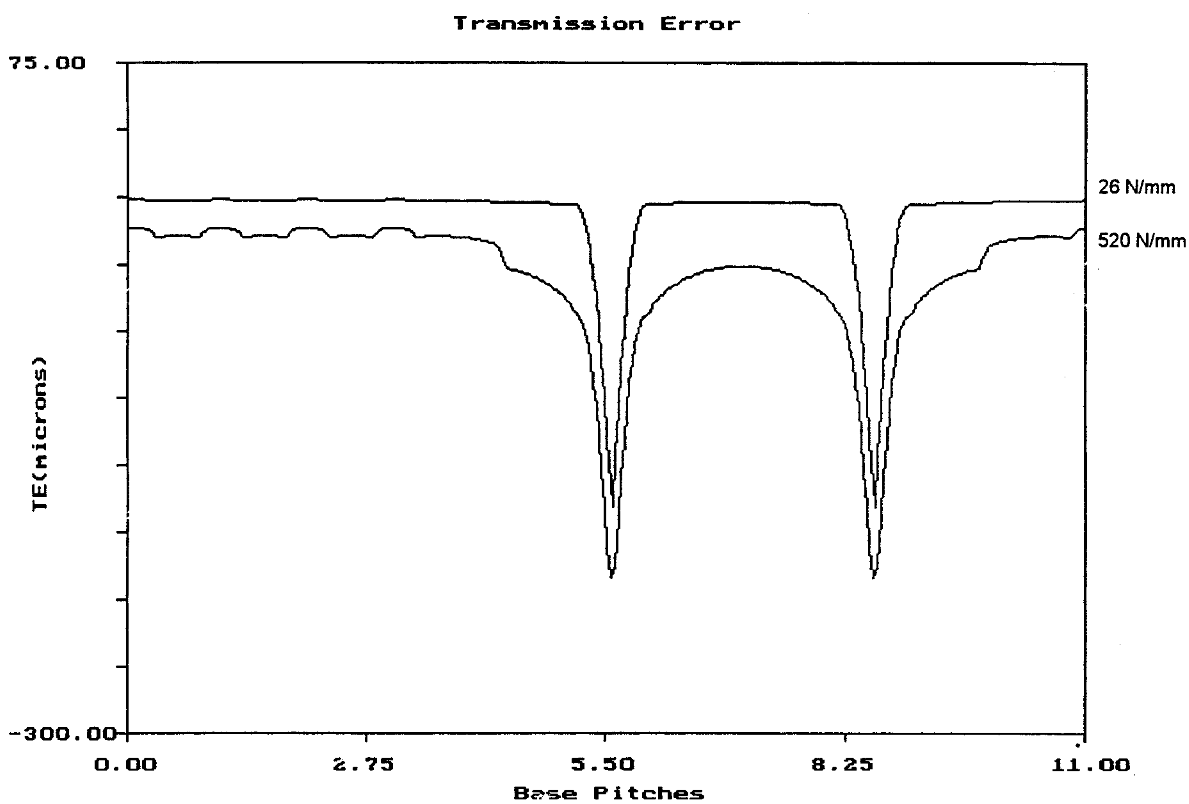

Fig. 7 Predicted transmission error of a gear pair with neighbouring teeth removed

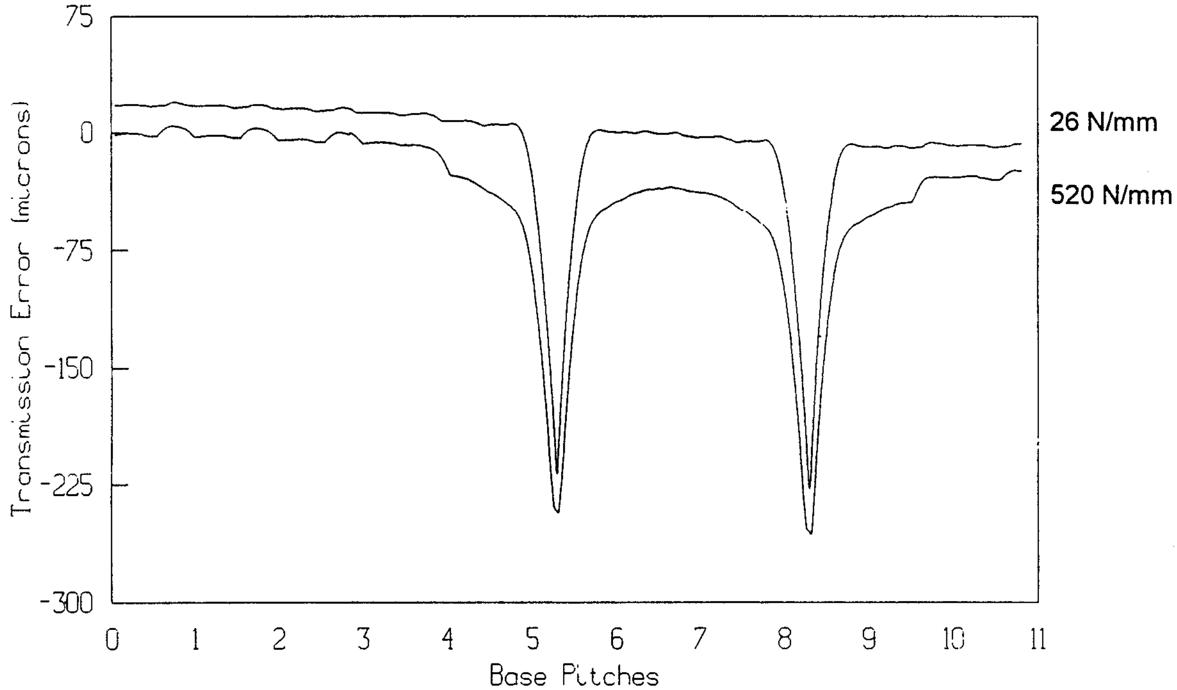

Fig. 8 Measured transmission error of a gear pair with neighbouring teeth removed

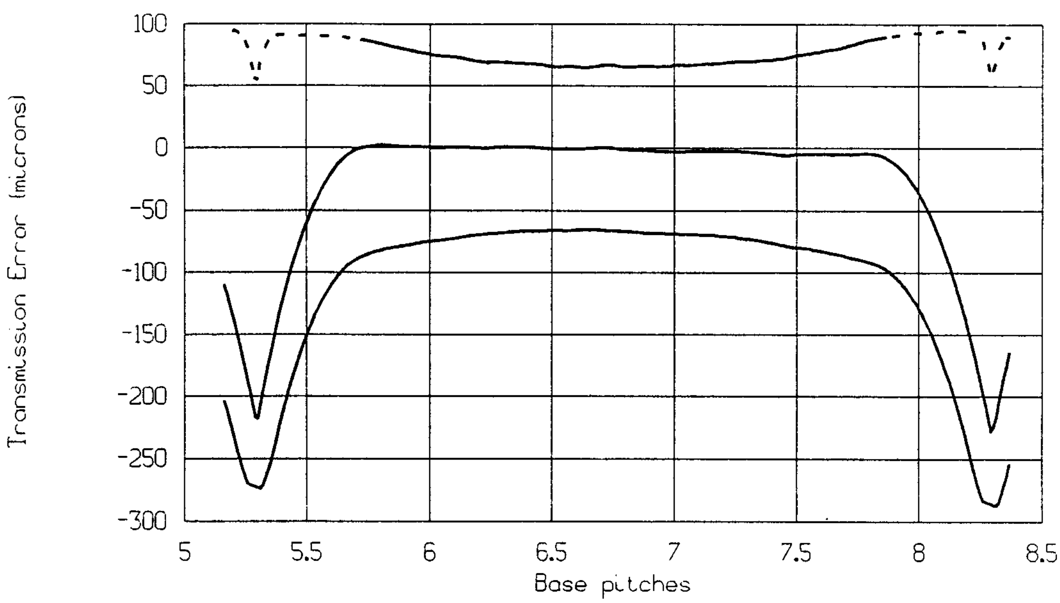

Fig. 9 Transmission error for one tooth pair (including bearing deflection) (loads 26–520 N/mm)

appearing as a spike are where the adjacent tooth pairs start to come into contact and there are two teeth pairs in contact so the stiffness starts to double.

The numerical values are as follows:

Load = 26 and 520 N/mm
$\delta_1 = 69.36\,\mu m$ (measured)
$\delta_2 = 50.26\,\mu m$ (measured)
$c_b = 15.85\,GPa$ [calculated from equation (3)]
$c_t = 12.93\,GPa$ [calculated from equation (4)]

When a parabola is fitted to the stiffness curve (Fig. 11), the stiffness at pitch point is found to be $\sim$14.25 GPa and at the ends of the path of contact 8.5 GPa.

The stiffness at the ends of the path of contact reduces to 60 per cent of the value at the centre. This is a slightly lower value than suggested before, which may be due to the longer teeth of the high contact ratio gears. If the

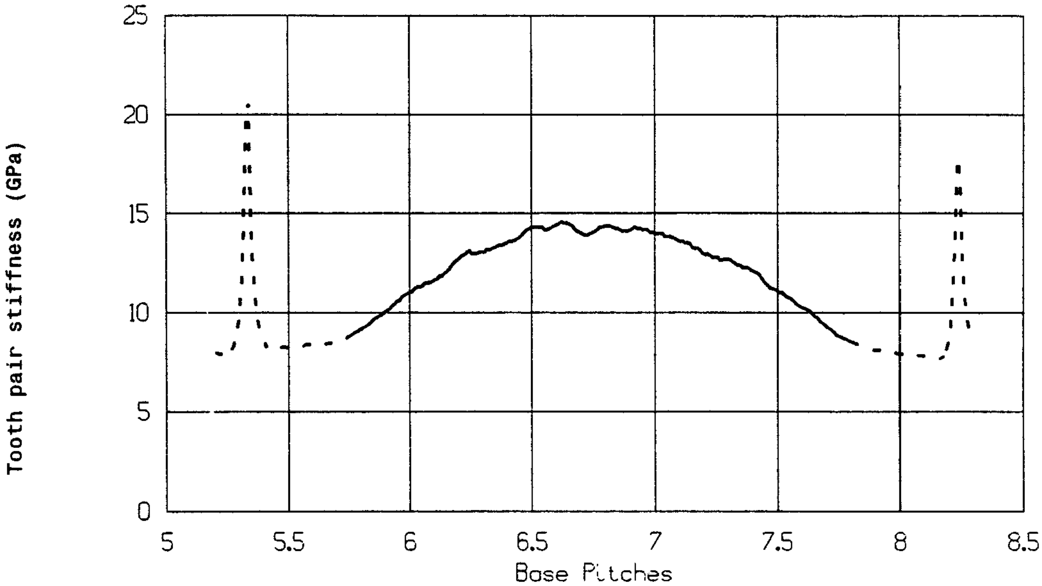

Fig. 10 Tooth pair stiffness (loads 26–520 N/mm)

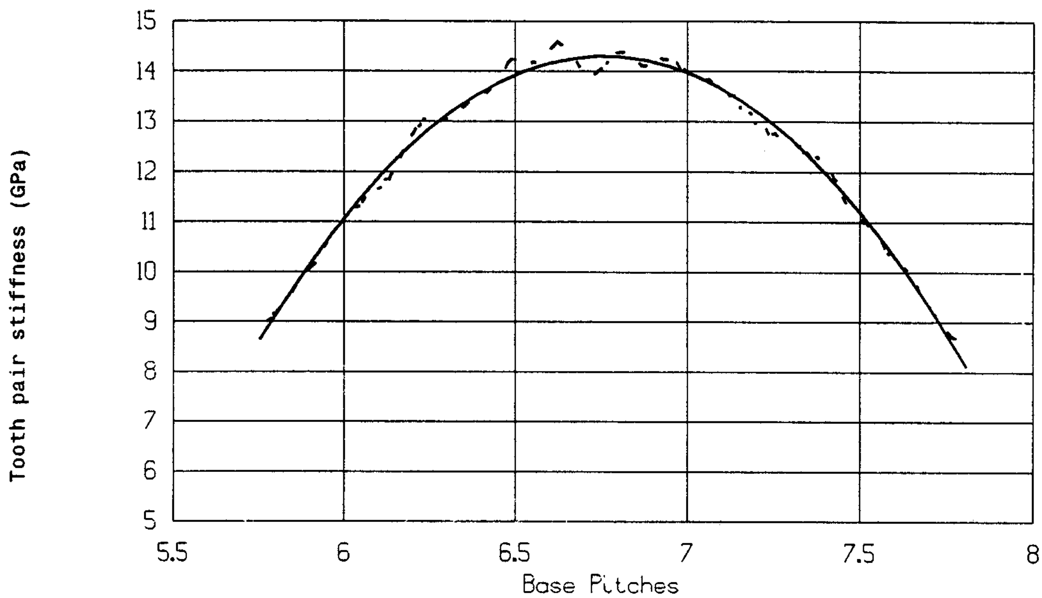

Fig. 11 Tooth pair stiffness with a best-fit parabola (loads 26–520 N/mm)

path of contact is modified for low contact ratio gears, with a contact ratio of 1.69 (by reducing the outside diameters of the gears), the values of stiffness at the ends of the path of contact are 10.25 GPa, giving a reduction to 71 per cent of the maximum value. The same procedure was carried out for the same set of gears for two higher loads and the results are shown in Figs 12 and 13.

The numerical values are as follows:

Load = 191 and 520 N/mm
$\delta_1 = 43.56\,\mu m$
$\delta_2 = 32.33\,\mu m$
$c_b = 15.59\,GPa$
$c_t = 14.65\,GPa$

The stiffness at the pitch point is $\sim$16.5 GPa and at the ends of the path of contact $\sim$9.5 GPa. The stiffness values are slightly greater for the higher load test, which may be due to the crowning on the gears. Under light load, the non-linear compression of the tooth surfaces as a result of Hertzian deflection is significant, whereas under the two heavy loads this effect is less pronounced.

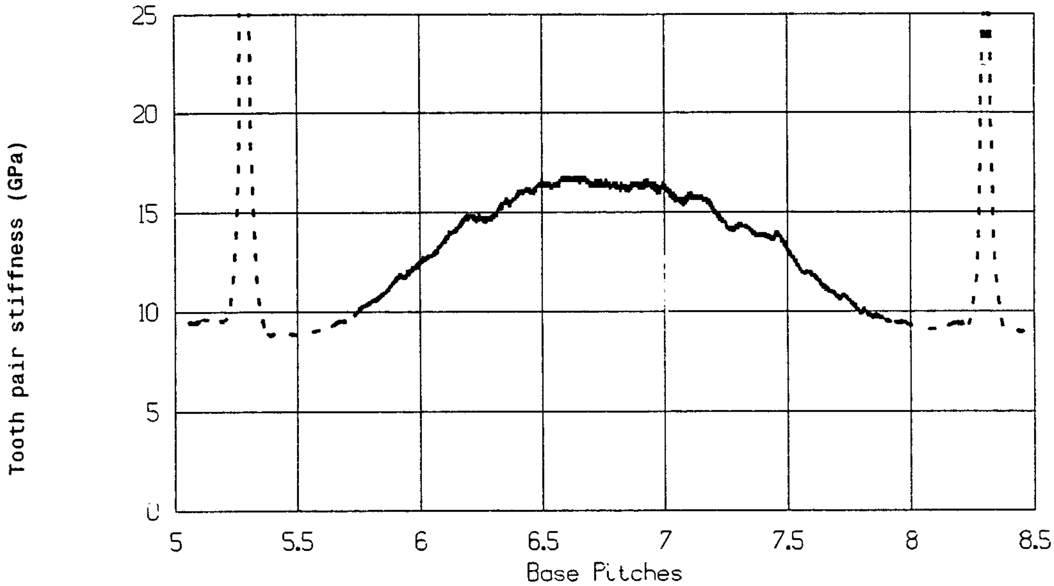

Fig. 12 Tooth pair stiffness (loads 191–520 N/mm)

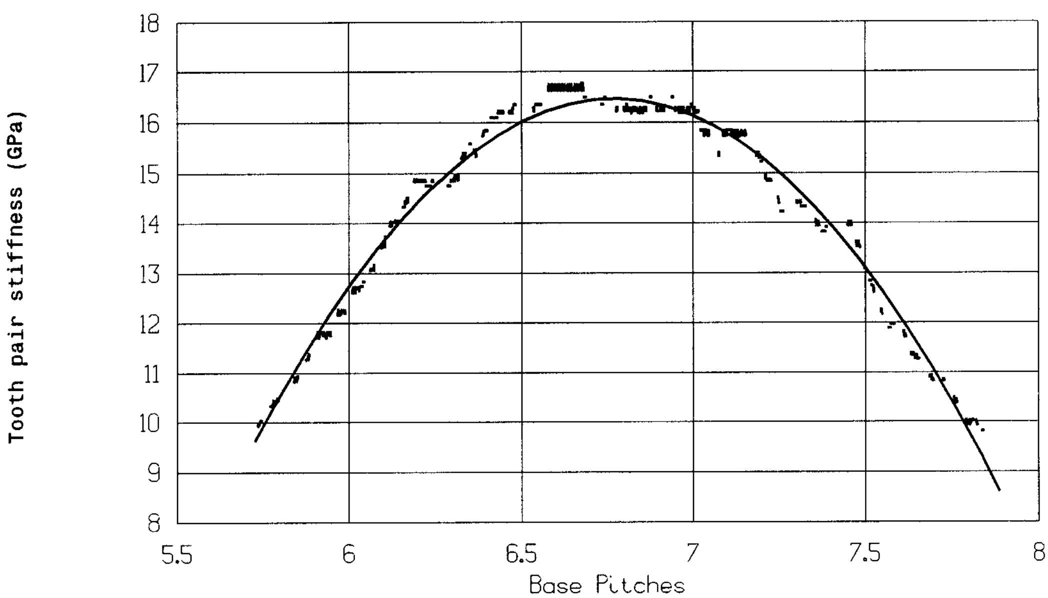

Fig. 13 Tooth pair stiffness with a best-fit parabola (loads 191–520 N/mm)

7 CONCLUSIONS

This method of measuring tooth stiffness has eliminated the uncertainty arising from other deflections in the test rig, and the stiffness values of 14.25 and 16.5 GPa obtained at the two load ranges appear to be logical when compared with other published values. They lie close to the centre of the range of values found in the survey by Steward, slightly below the value of approximately 18 GPa from the British Standard BS436 and slightly more than the design guideline value of 13.8 GPa mentioned in Section 2. Moreover, the difference between the values of 14.25 and 16.5 GPa is likely to be due to the greater Hertzian non-linearity effect at the lighter load, accentuated by the small amount of tooth crowning.

Additionally, the method underlines the value of the Harris map presentation, giving both alternating and mean values of transmission error, in their correct relative positions. Not only is it invaluable for vibration and noise studies, and for tooth relief design, but it is shown here also to provide a robust method for tooth stiffness measurement.

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